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INFINITELY ENTANGLED STATES

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For states in infinite dimensional Hilbert spaces entanglement quantities like the entanglement of distillation can become infinite. This leads naturally to the question, whether one system in such an infinitely entangled state can serve as a resource for tasks like the teleportation of arbitrarily many qubits. We show that appropriate states cannot be obtained by density operators in an infinite dimensional Hilbert space. However, using techniques for the description of infinitely many degrees of freedom from field theory and statistical mechanics, such states can nevertheless be constructed rigorously. We explore two related possibilities, namely an extended notion of algebras of observables, and the use of singular states on the algebra of bounded operators. As applications we construct the essentially unique infinite analogue of maximally entangled states, and the singular state used heuristically in the fundamental paper of Einstein, Rosen and Podolsky.

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Dedicated to the memory of Rob Clifton

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1 Introduction

Many of the concepts of entanglement theory were originally developed for quantum systems described in finite dimensional Hilbert spaces. This restriction is often justified, since we are usually only trying to coherently manipulate a small part of the system. On the other hand, a full description of almost any system, beginning with a single elementary particle, requires an infinite dimensional Hilbert space. Hence if one wants to discuss decoherence mechanisms arising from the coupling of the "qubit part" of the system with the remaining degrees of freedom, it is necessary to widen the context of entanglement theory to infinite dimensions. This is not difficult, since many of the basic notions, e.g. the definitions of entanglement measures, like the reduced von Neumann entropy or entanglement of formation, carry over almost unchanged, merely with finite sums replaced by infinite series. More serious are some technical problems arising from the fact that such entanglement measures can now become infinite, and are no longer continuous functions of the state. Luckily, as shown in recent work of Eisert et. al. [1], these problems can be tamed to a high degree, if one imposes some natural energy constraints on the systems.

In the present paper we look at some not-so-tame states, which should be considered as idealized descriptions of situations in which very much entanglement is available. For example, in the study of "entanglement assisted capacity" [2] one assumes that the communicating partners have an unlimited supply of shared maximally entangled singlets. In quantum information problems involving canonical variables it is easily seen that perfect operations can only be expected in the limit of an "infinitely squeezed" two mode gaussian state as entanglement resource (see also Section 6). But infinite entanglement is not only a desirable resource, it is also a natural property of some physical systems, such as the vacuum in quantum field theory (see [3, 4] and Section 5 below). Our aim is to show that one can analyze these situations by writing down bona fide states on suitably constructed systems.

It turns out that in order to do this we need to go one step beyond the standard Hilbert space formalism of quantum mechanics (even with infinite dimensional Hilbert spaces). This is completely analogous to other systems with infinitely many degrees of freedom, arising in quantum statistical mechanics (in the thermodynamic limit) and quantum field theory. For example, consider a translationally invariant, finite density equilibrium state of a gas. For such a state the probability for having only finitely many particles must obviously be zero. In contrast, a state given by density operators on Fock space implies a probability distribution of particle numbers, which are by definition finite numbers in $\mathbb{N}$. The expectation value of this distribution may be infinite. This means that in the limit of taking many ($m \to \infty$) equally prepared systems, the total number n of particles diverges faster than m, so that $n/m \to \infty$. However, the particle number of each single system remains finite.

Our first main result establishes a similar limitation for entanglement theory. Consider a state with infinite distillible entanglement. This means that when we take a large number m of copies of such system pairs, and try to obtain from these a large number n of nearly maximally entangled singlet pairs by local operations and classical communication, we can achieve a rate $n/m \to \infty$ as $m \to \infty$. In this asymptotic sense each pair contains infinitely many ebits of entanglement. But what does this mean for a single pair? Should we not get arbitrarily many singlets even out of this? According to Theorem 1 the answer is no, as long as we stay in standard quantum mechanics with bipartite states given by density operators on tensor product Hilbert spaces (of finite or infinite dimension).

However, if we follow the lead of statistical mechanics, and employ the methods for describing states with actually infinite particle number, we find a very natural framework to overcome this limitation: Here we can have a direct mathematical representation of the intuitive idea of "having infinitely many singlets". But we need to reconsider either the notion of states, allowing "singular" probability assignments on the space of quantum observables, which cannot be written in terms of density operators (cf. Section 4.2) or else allow more general observable algebras for Alice and Bob (cf. Section 4.3). These two approaches are closely related. We show in Section 5, how this extended framework leads to an essentially unique description of maximally entangled states of systems with infinitely many degrees of freedom. In the final Section 6 we discuss the "original EPR-state", i.e., a mathematically rigorous version of the singular state employed by Einstein, Rosen and Podolsky in their fundamental paper [5]. This Section builds on the work of Rob Clifton [6, 7, 8], who sadly died while this paper was in preparation. We dedicate it to his memory.

2 Density operators on infinite dimensional Hilbert space

We will start our discussion with a short look at entanglement properties of density operators on an infinite dimensional but separable[a] Hilbert space $\mathcal{H} \otimes \mathcal{H}$.

Most of the definitions of entanglement quantities carry over from the finite dimensional setting without essential change. Since we want to see how these quantities may diverge, let us look mainly at the smallest, the distillible entanglement. It is defined as the largest rate n/m of n nearly perfect singlets, which can be extracted from m pairs prepared in the given state by protocols involving local quantum operations and classical communication. The class of quantum operations is defined as in the finite dimensional case, by completely positive, trace preserving maps (in the Schrödinger picture). Effectively, distillation protocols for infinite dimensional systems can be built up by first projecting to a suitable finite dimensional subspace, and subsequently applying finite dimensional procedures to the result.

With this in mind we can easily construct *pure states* with infinite distillible entanglement. Let us consider vectors in Schmidt form, i.e.,

$$\Phi = \sum_n c_n \; e'_n \otimes e''_n \tag{1}$$

with orthonormal bases e'_n, e''_n, and positive numbers $c_n \geq 0$, and $\sum_n |c_n|^2 = 1$. The density operator of the restriction of this state to Alices's subsystem has eigenvalues c_n^2, and von Neumann entropy $-\sum_n c_n^2 \log_2(c_n^2)$, which we can take to be infinite ($c_n = 1/(Z(n+2)\log_2(n+2)^2)$ will do). We can distill this by using more and more of the dimensions as labelled by the bases e'_n, e''_n, and applying the known finite dimensional distillation procedures to this to get out arbitrary amount of entanglement per pair.

Once this is done, it is also easy to construct mixed states with large entanglement in the neighborhood of any state ρ, mixed or pure, separable or not. We only have to remember

[a]Another extension of this framework, namely to Hilbert spaces of uncountable dimension (i.e., unseparable ones in the topological sense) is not really interesting with regard to entanglement theory, since any density operator has separable support, i.e., it is zero on all but countably many dimensions.

that every state is essentially (i.e., up to errors of small probability) supported on a finite dimensional subspace. Therefore we can consider the mixture $\rho_\epsilon = (1-\epsilon)\rho + \epsilon\sigma$ with a small fraction of an infinitely entangled pure state σ, which is supported on those parts of Hilbert space, where ρ is nearly zero. Therefore distillation based on the support of σ will work for ρ_ϵ and produce arbitrarily large entanglement per ρ_ϵ pair, in spite of the constant reduction factor ϵ.

For the details of such arguments we refer to [7, 9]. The argument as given here does not quite show that states of infinite distillible entanglement are norm dense, but it certainly establishes the discontinuity of the function "distillible entanglement" with respect to the trace norm topology. This might appear to show that the approach to distillible entanglement based on finite dimensional systems is fundamentally flawed: If only finitely many dimensions out of the infinitely many providing a full description of the particle/system are used, might not the entanglement be misrepresented completely? Here it helps that states living on a far out subspace in Hilbert space usually also have large or infinite energy. For typical confined systems, the subspaces with bounded energy are finite dimensional, so if we assume a realistic a priori bound on the energy expectation of the states on the consideration, continuity can be restored [1].

3 Infinite one-copy entanglement

If the entanglement of formation of a state is infinite: how much of that entanglement can we get out? Since for pure states the distillible entanglement is the same as the entanglement of formation we know that given sufficiently many copies of the state, we can use a distillation process producing in the long run infinitely many nearly pure singlets per original entangled pair. But if the entanglement is infinite, might it not be possible to use only one copy of the state in the first place? In other words, are there states, which can be used as a *one time resource*, to teleport an arbitrary number of qubits?

We will now give a definition of such states. The extraction of entanglement will be described by a sequence of operations resulting in a pair of d-level systems with finite d. The extraction is successful, if this pair is in a nearly maximally entangled state, when one starts from the given input state. The overall operation is then given mathematically by a completely positive, trace preserving map $\mathcal{E}_d$. Of course, we must make sure that the extraction process does not generate entanglement. There are different ways of expressing this mathematically. For example, we could allow $\mathcal{E}_d$ to be composed of an arbitrary number of rounds of local quantum operations and classical communication ("LOCC operations"). We will also consider a much weaker, and much more easily verified condition, namely that $\mathcal{E}_d$ takes pure product states into states with positive partial transpose ("PPtPPT operations" for "pure product to positive partial transpose"). Of course, every LOCC channel is a PPtPPT channel.

The success is measured by the fidelity (overlap) of the output state $\mathcal{E}_d(\rho)$ with a fixed maximally entangled state on $\mathbb{C}^d \otimes \mathbb{C}^d$. By p_d we denote the projection onto this maximally entangled reference vector. Then a density operator ρ is said to have "infinite one-copy entanglement", if for any $\varepsilon > 0$ and any $d \in \mathbb{N}$ there is a PPtPPT channel $\mathcal{E}_d$ such that

$$\mathrm{tr}(\mathcal{E}_d(\rho)p_d) \geq 1 - \varepsilon \,. \tag{2}$$

Then we have the following Theorem, whose proof uses a distillation estimate of Rains [10] developed for the finite dimensional context.

Theorem 1 *For any sequence of PPtPPT channels $\mathcal{E}_d$, $d \in \mathbb{N}$, and for any fixed density operator ρ we have*

$$\lim_{d\to\infty} \mathrm{tr}(\mathcal{E}_d(\rho)p_d) = 0 \; . \tag{3}$$

In particular, no density operator with infinite one-copy entanglement exists.

Proof. Consider the operators A_d defined by

$$\mathrm{tr}(\rho A_d) = \mathrm{tr}(\mathcal{E}_d(\rho)p_d). \tag{4}$$

In order to verify that A_d exists, observe that $\mathcal{E}_d$, as a positive operator is automatically norm continuous. Hence the right hand side is a norm continuous linear functional on density matrices ρ. Since the set of bounded operators is the dual Banach space of the set of trace class operators [11, Theorem VI.26] such functionals are indeed of the form (4). We now have to show that, for every ρ, we have $\lim_d \mathrm{tr}(\rho A_d) = 0$, i.e., that $A_d \to 0$ in the weak*-topology of this dual Banach space.

Obviously, $0 \leq A_d \leq \mathbf{1}$, and by the Banach-Alaoglu Theorem [11, Theorem IV.21], this set is compact in the topology for which we want to show convergence. Hence the sequence has accumulation points and we only have to show that all accumulation points are zero. Let A_∞ denote such a point. Then it suffices to show that $\mathrm{tr}(\sigma A_\infty) = 0$ for all pure product states σ. Indeed, since $A_\infty \geq 0$, this condition forces $A_\infty \phi \otimes \psi = 0$ for all pairs of vectors ϕ, ψ, and hence $A_\infty = 0$, because such vectors span the tensor product Hilbert space.

On the other hand, our locality condition is strong enough to allow us to compute the limit directly for pure product states σ. We claim that

$$\mathrm{tr}(\sigma A_d) = \mathrm{tr}(\mathcal{E}_d(\sigma)p_d) = \mathrm{tr}(\mathcal{E}_d(\sigma)^{T_2} p_d^{T_2}) \leq \|p_d^{T_2}\| = 1/d$$

Here we denote by X^{T_2} the partial transposition with respect to the second tensor factor, of an operator X on the finite dimensional space $\mathbb{C}^d \otimes \mathbb{C}^d$, and use that this operation is unitary with respect to the Hilbert-Schmidt scalar product $\langle X, Y\rangle_{HS} = \mathrm{tr}(X^*Y)$. By assumption, $\mathcal{E}_d(\sigma)^{T_2} \geq 0$, and since partial transposition preserves the trace, $\mathcal{E}_d(\sigma)^{T_2}$ is even a density operator. Hence the expectation value of $p_d^{T_2}$ in this state is bounded by the norm of this operator. But it is easily verified that $p_d^{T_2}$ is just $(1/d)$ times the unitary operator exchanging the two tensor factors. Hence its norm is $(1/d)$. Taking the limit of this estimate along a sub-sequence of A_d converging to A_∞, we find $\mathrm{tr}(\sigma A_\infty) = 0$. □

4 Singular states and infinitely many degrees of freedom

In this section we will show how to construct a system of infinitely many singlets. It is clear from Theorem 1 that not all of the well-known features of the finite situation will carry over. Nevertheless, we will stay as closely as possible to the standard constructions trying to pretend that ∞ is finite, and work out the necessary modifications as we go along.

Before we proceed in this direction let us add some short remarks about the physical significance of this construction. Each "real" system consists of course only of finitely many particles and can be described therefore by a finite (although arbitrarily large) tensor product.

Nevertheless the idealization of infinitely many particles has often proved to be very useful; e.g. in the study of phase transitions. The system we are going to construct – infinitely many singlets – should be regarded in the same way as an idealized limiting case of an "inexhaustible" entanglement resource.

4.1 Von Neumann's incomplete infinite tensor product of Hilbert spaces

The first difficulty we encounter is the construction of Hilbert spaces for Alice's and Bob's subsystem, respectively, which should be the *infinite tensor power* $(\mathbb{C}^2)^{\otimes\infty}$ of the one qubit space $\mathbb{C}^2$. Let us recall the definition of a tensor product: it is a Hilbert space generated by linear combination and norm limits from basic vectors written as $\Phi = \bigotimes_{j=1}^{\infty} \phi_j$, where ϕ_j is a vector in the jth tensor factor. All we need to know to construct the tensor product as the completion of formal linear combinations of such vectors are their scalar products, which are, by definition,

$$\left\langle \bigotimes_{j=1}^{\infty} \phi_j, \bigotimes_{j=1}^{\infty} \psi_j \right\rangle = \prod_{j=1}^{\infty} \langle \phi_j, \psi_j \rangle . \tag{5}$$

The problem lies in this infinite product, which clearly need not converge for arbitrary choice of vectors ϕ_j, ψ_j. A well-known way out of this dilemma, known as *von Neumann's incomplete tensor product* [12] is to restrict the possible sequences of vectors $\phi_1, \phi_2, \dots$ in the basic product vectors: for each tensor factor, one picks a reference unit vector χ_j, and only sequences are allowed for which $\phi_j = \chi_j$ holds for all but a finite number of indices. Evidently, if this property holds for both the ϕ_j and the ψ_j the product in (5) contains only a finite number of factors $\neq 1$, and converges. By taking norm limits of such vectors we see that also product vectors for which $\sum_{j=1}^{\infty} \|\phi_j - \chi_j\| < \infty$ are included in the infinite product Hilbert space. However, the choice of reference vectors χ_j necessarily breaks the full unitary symmetry of the factors, as far as asymptotic properties for $j \to \infty$ are concerned. For the case at hand, i.e., qubit systems, let us choose, for definiteness, the "spin up" vector as χ_j for every j, and denote the resulting space by $\mathcal{H}_\infty$.

An important observation about this construction is that all observables of finite tensor product subsystems act as operators on this infinite tensor product space. In fact, any operator $\bigotimes_{j=1}^{\infty} A_j$ makes sense on the incomplete tensor product, as long as $A_j = \mathbb{1}$ for all but finitely many indices. The algebra of such operators is known as the algebra of local observables. It has the structure of a *-algebra, and its closure in operator norm is called *quasi-local algebra* [13].

Let us take the space $\mathcal{H}_\infty$ as Alice's and Bob's Hilbert space. Then each of them holds infinitely many qubits, and we can discuss the entanglement contained in a density operator on $\mathcal{H}_\infty \otimes \mathcal{H}_\infty$. Clearly, there is no general upper bound to this entanglement, since we can take a maximally entangled state on the first $M < \infty$ factors, complemented by infinitely many spin-up product states on the remaining qubit pairs. But for any fixed density operator the entanglement is limited: for measurements on qubit pairs with sufficiently large j we always get nearly the same expectations as for two uncorrelated spin-up qubits (or whatever the reference states χ_j dictate). This is just another instance of Theorem 1: there is no density operator describing infinitely many singlets.

4.2 Singular states

However, can we not take the limit of states with growing entanglement? To be specific, let Φ_M denote the vector which is a product of singlet states for the first M qubit pairs, and a spin-up product for the remaining ones. These vectors do not converge in $\mathcal{H}_\infty \otimes \mathcal{H}_\infty$, but that need not concern us, if we are only interested in expectation values: for all local observables A (observables depending on only finitely many qubits), the limit

$$\omega(A) = \lim_M \langle \Phi_M, A\Phi_M \rangle \tag{6}$$

exists. Thereby we get an expectation value functional for all quasi-local observables, and by the Hahn-Banach Theorem (see e.g. [11, Theorem III.6]), we can extend this expectation value functional to all bounded operators on $\mathcal{H}_\infty \otimes \mathcal{H}_\infty$. The extended functional ω has all the properties required by the statistical interpretation of quantum mechanics: linearity in A, $\omega(A) \geq 0$ for positive A, and $\omega(\mathbb{1}) = 1$. In the terminology of the theory of operator algebras, it is a *state* on the algebra of all bounded operators. By construction, ω describes maximal entanglement for *any* finite collection of qubit pairs, so it is truly a state of infinitely many singlets.

How does this match with Theorem 1? The crucial point is that that Theorem only speaks of states given by the trace with a density operator, i.e., of functionals of the form $\omega_\rho(A) = \mathrm{tr}(\rho A)$. Such states are called "normal". But there is no density operator for ω: this is a *singular state* on the algebra of bounded operators.

Singular states are not that unusual in quantum mechanics, although they can only be "constructed" by an invocation the Axiom of Choice, usually through the Hahn-Banach Theorem.[b] For example, we can think of a non-relativistic particle localized at a sharp point, as witnessed by the expectations of all continuous functions of position. Extending from this algebra to all bounded operators, we get a singular state with sharp position,[c] but "infinite momentum", i.e., the probability assigned to finding the momentum in any given finite interval is zero [15]. This shows that the probability measure on the momentum space induced by such a state is only finitely additive, but not σ-additive. This is typical for singular states.

More practical situations involving singular states arise in all systems with infinitely many degrees of freedom, as in quantum field theory and in statistical mechanics in the thermodynamic limit[16, 13]. For example, the equilibrium state of a free Bose gas in infinite space at finite density and temperature is singular with respect to Fock space because the probability for finding only a finite number of particles in such a state is zero. In all these cases, one is primarily interested in the expectations of certain meaningful observables (e.g., local observables), and the wilder aspects of singular states are connected only to the extension of the state to *all* bounded operators. Therefore it is a good strategy to focus on the state as an expectation functional only on the "good" observables.

4.3 Local observable algebras

If we want to represent a situation with infinitely many singlets, an obvious approach is to take again von Neumann's incomplete tensor product, but this time the infinite tensor product of

[b] Other constructions based on the Axiom of Choice are the application of invariant means, e.g., when averaging expectation values over all translations, or algebraic constructions using maximal ideals. For an application in von Neumann style measurement theory of continuous spectra, see [14]

[c] This is not related to improper eigenkets of position, which do not yield normalized states

pairs rather than single qubits, with the singlet vector chosen as the reference vector χ_j for every pair. We denote this space by $\mathcal{H}_{\infty\infty}$, and by $\Omega \in \mathcal{H}_{\infty\infty}$ the infinite tensor product of singlet vectors. Clearly, this is a normal state (with density operator $|\Omega\rangle\langle\Omega|$), and we seem to have gotten around Theorem 1 after all.

However, the problem is now to identify the Hilbert spaces of Alice and Bob as tensor factors of $\mathcal{H}_{\infty\infty}$. To be sure, the observables measurable by Alice and Bob, respectively, are easily identified. For example, the σ_x-Pauli matrix for Alice's 137^{th} particle is a well defined operator on $\mathcal{H}_{\infty\infty}$. Alice's observable algebra $\mathcal{A}$ is generated by the collection of all Alice observables for each pair. Bob's observable algebra $\mathcal{B}$ is generated similarly, and together they generate the local algebra of the pair system. Moreover, the two observable algebras commute elementwise. This is just what we expect from the usual setup, when the total Hilbert space is $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, and Alice's and Bob's observable algebras are $\mathcal{A} = \mathcal{B}(\mathcal{H}_A) \otimes \mathbb{1}_B$ and $\mathcal{B} = \mathbb{1}_A \otimes \mathcal{B}(\mathcal{H}_B)$.

However, the $\mathcal{A}$ and $\mathcal{B}$ constructed above are definitely not of this form, so $\mathcal{H}_{\infty\infty}$ has no corresponding decomposition as $\mathcal{H}_A \otimes \mathcal{H}_B$. The most direct way of seeing this is to note that $\mathcal{H}_{\infty\infty}$ contains no product vectors describing an uncorrelated preparation of the two subsystems. If we move to qubit pairs with sufficiently high index, then by construction of the incomplete tensor product, *every* vector in $\mathcal{H}_{\infty\infty}$ will be close to the singlet vector, and in particular, will violate Bell's inequality nearly maximally (see also Section 5.2).

Hence we arrive at the following generalized notion of bipartite states, generalizing the finite dimensional one: Alice's and Bob's subsystems are identified by their respective observable algebras $\mathcal{A}$ and $\mathcal{B}$. We postpone the discussion of the precise technical properties of these algebras. What is important is, on the one hand, that these algebras are part of a larger system, so they are both subalgebras of a larger algebra, typically the algebra $\mathcal{B}(\mathcal{H})$ of bounded operators on some Hilbert space. This allows us to consider products and correlations between the two algebras. On the other hand, each measurement Alice chooses must be compatible with each one chosen by Bob. This requires that $\mathcal{A}$ and $\mathcal{B}$ commute elementwise. A *bipartite state* is then simply a state on the algebra containing both $\mathcal{A}$ and $\mathcal{B}$.

We can then describe the two ways out of the NoGo-Theorem: on the one hand we can allow more general states than density matrices, but on the other hand we can also consider more general observable algebras. In the examples we will discuss, the algebra containing $\mathcal{A}$ and $\mathcal{B}$ will in fact be of the form $\mathcal{B}(\mathcal{H})$, and the states will be given by density matrices on $\mathcal{H}$. So both strategies can be successful by themselves.

4.4 Some basic facts about operator algebras

The possibility of going either to singular states or to extended observable algebras is typical of the duality of states and observables in quantum mechanics. There are many contexts, where it is useful to extend either the set of states or the set of observables by idealized elements, usually obtained by some limit. However, these two idealizations may not be compatible [15]. There are two types of operator algebras which differ precisely in the strength of the limit procedures under which they are closed [13, 17].

On the one hand there are *C*-algebras*, which are isomorphic to norm and adjoint closed algebras of operators on a Hilbert space. Norm limits are quite restrictive, so some operations are not possible in this framework. In particular, the spectral projections of an hermitian element of the algebra often do not lie again in the algebra (although all continuous functions will). Therefore, it is often useful to extend the algebra by all elements obtained as weak limits (meaning that all matrix elements converge). In such *von Neumann algebras* the spectral theorem holds. Moreover, the limit of an increasing but bounded sequence of elements always converges in the algebra. For these algebras the distinction between normal and singular states becomes relevant. The normal states are simply those for which such increasing limits converge, and at the same time those which can be represented by a density operator in the ambient Hilbert space.

A basic operation for von Neumann algebras is the formation of the *commutant*: for any set $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ closed under the adjoint operation, we define its commutant as the von Neumann algebra

$$\mathcal{M}' = \left\{ X \in \mathcal{B}(\mathcal{H}) \,\middle|\, \forall M \in \mathcal{M}\ [M, X] = 0 \right\} . \tag{7}$$

Then the Bicommutant Theorem [17] states that $\mathcal{M}'' = (\mathcal{M}')'$ is the smallest von Neumann algebra containing $\mathcal{M}$. In particular, when $\mathcal{M}$ is already an algebra, $\mathcal{M}''$ is the weak closure of $\mathcal{M}$. Von Neumann algebras are characterized by the property $\mathcal{M}'' = \mathcal{M}$. A von Neumann algebra $\mathcal{M}$ with the property that its only elements commuting with all others are the multiples of the identity (i.e., $\mathcal{M}' \cap \mathcal{M}'' = \mathbb{C}\mathbf{1}$) is called a *factor*.

It might seem that the two ways out of the NoGo-Theorem indicated at the end of the previous section are opposite to each other, but in fact they are closely related. For if ω is a state on a C*-algebra $\mathcal{C} \supset \mathcal{A} \cup \mathcal{B}$, we can associate with it a Hilbert space $\mathcal{H}_\omega$, a representation $\pi_\omega : \mathcal{C} \to \mathcal{B}(\mathcal{H})$, and a unit vector $\Omega \in \mathcal{H}_\omega$, such that $\omega(C) = \langle \Omega, \pi_\omega(C)\Omega \rangle$, and such that the vectors $\pi_\omega(C)\Omega$ are dense in $\mathcal{H}_\omega$. This is called the Gelfand-Naimark-Segal (GNS)-construction [13]. Clearly, the given state ω is given by a density operator (namely $|\Omega\rangle\langle\Omega|$) in this new representation and the algebra can naturally be extended to the weak closure $\pi_\omega(C)''$. The commutativity of two subalgebras is preserved by the weak closure, so the normal state $|\Omega\rangle\langle\Omega|$, and the two commuting von Neumann subalgebas $\pi_\omega(\mathcal{A})''$ and $\pi_\omega(\mathcal{B})''$ are again a bipartite system, which describes essentially the same situation. The only difference is that some additional idealized observables arise from the weak closure operations, and that some observables in $\mathcal{C}$ (those with $C \geq 0$ but $\omega(C) = 0$) are represented by zero in π_ω.

We remark that von Neumann's incomplete infinite tensor product of Hilbert spaces can be seen as a special case of the GNS-construction: The infinite tensor product of C*-algebras $\bigotimes_i \mathcal{A}_i$ is well-defined (see [13, Sec 2.6] for precise conditions), essentially by taking the norm completion of the algebra of *local observables* $\bigotimes_i A_i$, with all but finitely many factors $A_i \in \mathcal{A}_i$ equal to $\mathbf{1}_i$. On this algebra the infinite tensor product of states is well-defined, and we get the incomplete tensor product as the GNS-Hilbert space of the algebra $\bigotimes_i \mathcal{B}(\mathcal{H}_i)$ with respect to a the pure product state defined by the reference vectors χ_i.

5 Von Neumann algebras with maximal entanglement

5.1 Characterization and basic properties

Let us analyze the example given in the last section: the bipartite state obtained from the incomplete tensor product of singlets in $\mathcal{H}_{\infty\infty}$. We take as Alice's observable algebra $\mathcal{A}$ the von Neumann algebra generated by all local Alice operators (and analogously for $\mathcal{B}$ob). The bipartite state on these algebras, given by the reference vector $\bigotimes_i \chi_i$, then has the following properties

ME 1 $\mathcal{A}$ and $\mathcal{B}$ together *generate* $\mathcal{B}(\mathcal{H})$ as a von Neumann algebra, so there are no additional observables of the system beyond those measurable by Alice and Bob.

ME 2 $\mathcal{A}$ and $\mathcal{B}$ are *maximal* with respect to mutual commutativity. (i.e., $\mathcal{A} = \mathcal{B}'$ and $\mathcal{B} = \mathcal{A}'$)

ME 3 The overall state is *pure*, i.e., given by a vector $\Omega \in \mathcal{H}$,

ME 4 The restriction of this state to either subsystem is a *trace*, so $\omega(A_1 A_2) = \omega(A_2 A_1)$, for $A_1, A_2 \in \mathcal{A}$ (together with "ω is a state" that defines "trace").

ME 5 $\mathcal{A}$ is *hyperfinite*, i.e., it is the weak closure of an increasing family of finite dimensional algebras.

These properties, except perhaps ME 2 (see [18]) are immediately clear from the construction, and the properties of the respective local observables. They are also true for finite dimensional maximally entangled states on $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, $\mathcal{A} = \mathcal{B}(\mathcal{H}_A) \otimes \mathbb{1}$, and $\mathcal{B} = \mathbb{1} \otimes \mathcal{B}(\mathcal{H}_B)$. This justifies calling this particular bipartite system *maximally entangled*, as well.

There are many free parameters in this construction. For example, we could take arbitrary dimensions $d_i < \infty$ for the i^{th} pair. However, all these possibilities lead to the same maximally entangled system:

Theorem 2 *All bipartite states on infinite dimensional systems satisfying conditions* ME 1 - ME 5 *above are unitarily isomorphic.*

Proof. (Sketch). We first remark that $\mathcal{A}$ has to be a *factor*, i.e., $\mathcal{A} \cap \mathcal{A}' = \mathbb{C}\mathbb{1}$. Indeed, using ME 1 and ME 2, we get $\mathcal{A} \cap \mathcal{A}' = \mathcal{B}' \cap \mathcal{A}' = (\mathcal{B} \cup \mathcal{A})' = \mathcal{B}(\mathcal{H})' = \mathbb{C}\mathbb{1}$.

Now consider the support projection $S \in \mathcal{A}$ of the restriction of the state to $\mathcal{A}$. Thus $\mathbb{1} - S$ is the largest projection in $\mathcal{A}$ with vanishing expectation. Suppose that this projection does not lie in the center of $\mathcal{A}$, i.e., there is an $A \in \mathcal{A}$ such that $AS \neq SA$. Let $X = (\mathbb{1} - S)AS$, which must then be nonzero, as $AS - SA = ((\mathbb{1} - S) + S)(AS - SA) = X - SA(\mathbb{1} - S)$. Then using the trace property we get $\omega(X^*X) = \omega(XX^*) \leq \|A\|^2 \omega(\mathbb{1} - S) = 0$, which implies that the support projection of X^*X has vanishing expectation. But since $X^*X \leq \|A\|^2 S$, this contradicts the maximality of $(\mathbb{1} - S)$. It follows that S lies in the center of $\mathcal{A}$ and that $S = \mathbb{1}$, because $\mathcal{A}$ is a factor. To summarize this argument, ω must be *faithful*, in the sense that $A \in \mathcal{A}$, $A \geq 0$, and $\omega(A) = 0$ imply $A = 0$.

Now consider the subspace spanned by all vectors of the form $A\Omega$, with $A \in \mathcal{A}$. This subspace is invariant under $\mathcal{A}$, so its orthogonal projection is in $\mathcal{A}' = \mathcal{B}$. But since $(\mathbb{1} - P)$ obviously has vanishing expectation, the previous arguments, applied to $\mathcal{B}$ imply that $P = \mathbb{1}$. This is to say that $\mathcal{A}\Omega$ is dense in $\mathcal{H}$ or, in the jargon of operator algebras, that Ω is *cyclic* for $\mathcal{A}$. Thus $\mathcal{H}$ is unitarily equivalent to the GNS-Hilbert space of ω restricted to $\mathcal{A}$, and the form of $\mathcal{B} = \mathcal{A}'$ is completely determined by this statement. Now a factor admits at most one trace state, so ω is uniquely determined by the isomorphism type of $\mathcal{A}$ as a von Neumann algebra, and it remains to show that $\mathcal{A}$ is uniquely determined by the above conditions.

$\mathcal{A}$ is a factor admitting a faithful normal trace state, so it is a "type II_1-factor" in von Neumann's classification. It is also hyperfinite, so we can invoke a deep result of Alain Connes [19] stating that such a factor is uniquely determined up to isomorphism. □

For the rest of this section we will study further properties of this unique maximally entangled state of infinite entanglement. The items ME 6, ME 7 below are clear from the above proof. ME 8 follows by splitting the infinite tensor product either into a finite product and an infinite tail, or into factors with even and odd labels, respectively. ME 9 - ME 11 are treated in separate subsections as indicated.

ME 6 $\mathcal{A}$ and $\mathcal{B}$ are factors: $\mathcal{A} \cap \mathcal{A}' = \mathbb{C}\mathbb{1}$.

ME 7 $\mathcal{A}\Omega$ and $\mathcal{B}\Omega$ are dense[d] in $\mathcal{H}$.

ME 8 The state contains infinite one-shot entanglement, which is not diminished by extracting entanglement. Moreover, it is unitarily isomorphic to two copies of itself.

ME 9 Every density operator on $\mathcal{H}$ maximally violates the Bell-CHSH inequality (see Section 5.2).

ME 10 The generalized Schmidt spectrum of Ω is flat (see Section 5.3).

ME 11 Every $A \in \mathcal{A}$ is completely correlated with a "double" $B \in \mathcal{B}$. (see Section 5.4).

5.2 Characterization by violations of Bell's inequalities

If we look at systems consisting of two qubits, maximally entangled states can be characterized in terms of maximal violations of Bell-inequalities. It is natural to ask, whether something similar holds for the infinite dimensional setting introduced in Section 5. To answer this question consider again a bipartite state ω on an algebra containing two mutually commuting algebras $\mathcal{A}, \mathcal{B}$ describing Alice's and Bob's observables, respectively. We define the Bell correlations with respect to $\mathcal{A}$ and $\mathcal{B}$ in ω as

$$\beta(\omega) = \frac{1}{2} \sup \omega(A_1(B_1 + B_2) + A_2(B_1 - B_2)), \tag{8}$$

where the supremum is taken over all selfadjoint $A_i \in \mathcal{A}$, $B_j \in \mathcal{B}$ satisfying $-\mathbb{1} \leq A_i \leq \mathbb{1}$, $-\mathbb{1} \leq B_j \leq \mathbb{1}$, for $i, j = 1, 2$. In other words A_1, A_2 and B_1, B_2 are (appropriately bounded) observables measurable by Alice respectively Bob. Of course, a classically correlated (separable) state, or any other state consistent with a local hidden variable model [21] satisfies the Bell-CHSH-inequality $\beta(\omega) \leq 1$.

Exactly as in the standard case, we can show Cirelson's inequality [22, 23, 24] bounding the quantum violations of the inequality as

$$\beta(\omega) \leq \sqrt{2}. \tag{9}$$

If the upper bound $\sqrt{2}$ is attained we speak of a *maximal violation* of Bell's inequality.

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[d]This property has a close analogy within quantum field theory: Due to the famous Reeh-Schlieder Theorem [20] the vacuum state of a quantum field theory is (under quite general conditions) cyclic with respect to each local observable algebra.

It is clear that the maximally entangled state described above does saturate this bound: In the infinite tensor product construction of $\mathcal{H} = \mathcal{H}_{\infty\infty}$ we only need to take observables A_i, B_i from the first tensor factor. But we could also have chosen similar observables $A_{i,k}, B_{i,k}$ $(i = 1, 2)$ for the k^{th} qubit pair. Let us denote by

$$T_k = A_{1,k}(B_{1,k} + B_{2,k}) + A_{2,k}(B_{1,k} - B_{2,k}) \tag{10}$$

the "test operator" for the k^{th} qubit pair, whose expectation enters the Bell-CHSH-inequality. Then for a dense set of vectors $\phi \in \mathcal{H}$, namely for those differing from the reference vector in only finitely many positions, we get $\langle \phi, T_k \phi \rangle = \sqrt{2}$ for all sufficiently large k. Since the norms $\|T_k\|$ are uniformly bounded, a simple 3ε-argument shows that $\lim_{k\to\infty} \langle \phi, T_k \phi \rangle = \sqrt{2}$ for all $\phi \in \mathcal{H}_{\infty\infty}$. By taking mixtures we find

$$\lim_{k\to\infty} \operatorname{tr}(\rho T_k) = \sqrt{2} \tag{11}$$

for all density operators ρ on $\mathcal{H}_{\infty\infty}$.

This property is clearly impossible in the finite dimensional case: any product state would violate it. This clarifies the statement in Section 4.3 that $\mathcal{H}_{\infty\infty}$ is in no way a tensor product of Hilbert spaces for Alice and Bob. Of course, we can simply *define* a product state on the algebra of local operators, and then extend it by the Hahn-Banach Theorem to all operators on $\mathcal{B}(\mathcal{H}_{\infty\infty})$. However, just as the reference state of infinitely many singlets is a singular state on $\mathcal{B}(\mathcal{H}_\infty \otimes \mathcal{H}_\infty)$, any product state will necessarily be singular on $\mathcal{B}(\mathcal{H}_{\infty\infty})$.

It is interesting that bipartite states with property (11) naturally arise in quantum field theory, with $\mathcal{A}$ and $\mathcal{B}$ the algebras of observables measurable in two causally disjoint (but tangent) spacetime regions.[e] This is true under axiomatic assumptions on the structure of local algebras, believed to hold in any free or interacting theory. The only thing that enters is indeed the structure of the local von Neumann algebras, as shown by the following Theorem [3, 4, 23]. Again the maximally entangled state plays a key role.

Theorem 3 ([4]) *Let $\mathcal{A}, \mathcal{B} \subset \mathcal{B}(\mathcal{H})$ be mutually commuting von Neumann algebras acting on a separable Hilbert space $\mathcal{H}$. Then the following are equivalent:*

(i) For some density operator ρ, which has no zero eigenvalues, we have $\beta(\rho) = \sqrt{2}$.

(ii) For every density operator ρ on $\mathcal{H}$ we have $\beta(\rho) = \sqrt{2}$.

(iii) There is a set T_k of test operators formed from $\mathcal{A}$ and $\mathcal{B}$ such that (11) holds for all density operators ρ.

(iv) There is a unitary isomorphism under which

$$\mathcal{H} = \mathcal{H}_{\infty\infty} \otimes \widetilde{\mathcal{H}}\,, \quad \mathcal{A} = \mathcal{A}_1 \otimes \widetilde{\mathcal{A}}\,, \quad \mathcal{B} = \mathcal{B}_1 \otimes \widetilde{\mathcal{B}}\,,$$

$\mathcal{A}_1, \mathcal{B}_1 \subset \mathcal{B}(\mathcal{H}_{\infty\infty})$ are the algebras of Theorem 2, and $\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}} \subset \mathcal{B}(\widetilde{\mathcal{H}})$ are other von Neumann algebras.

In other words, the maximal violation of Bell's inequalities for *all* normal states implies that the bipartite system is precisely the maximal entangled state, plus some additional degrees of freedom $(\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}})$, which do not contribute to the violation of Bell inequalities.

[e] Note however that it is crucial here that both localization regions are really *tangent to one another*. If they are located at a nonvanishing distance d the situation changes drastically and it can be shown that Bell correlations vanish exponentially fast with d [25].

5.3 Schmidt decomposition and modular theory

The Schmidt decomposition is a key technique for analyzing bipartite pure states in the standard framework. It represents an arbitrary vector $\Omega \in \mathcal{H}_A \otimes \mathcal{H}_B$ as

$$\Omega = \sum_{\alpha} c_\alpha \, e_\alpha \otimes f_\beta \ , \tag{12}$$

where the $c_\alpha > 0$ are positive constants, and $\{e_\alpha\} \subset \mathcal{H}_A$ and $\{f_\alpha\} \subset \mathcal{H}_B$ are orthonormal systems.

Its analog in the context of von Neumann algebras is a highly developed theory with many applications in quantum field theory and statistical mechanics, known as the *modular theory* of Tomita and Takesaki [26]. We recommend Chapter 2.5 in [13] for an excellent exposition, and only outline some ideas and indicate the connection to the Schmidt decomposition.

Throughout this subsection, we will assume that $\mathcal{A}, \mathcal{B} \subset \mathcal{B}(\mathcal{H})$ are von Neumann algebras, and $\Omega \in \mathcal{H}$ is a unit vector, such that the properties ME 2, ME 3, and ME 7 of Section 5.1 hold. As in the case of the usual Schmidt decomposition the essential information is already contained in the restriction of the given state to the subalgebra $\mathcal{A}$, i.e., by the linear functional $\omega(A) = \langle \Omega, A\Omega \rangle$. Indeed, the Hilbert space and the cyclic vector Ω (cf. ME 7) satisfy precisely the conditions for the GNS-representation, which is unique up to unitary equivalence. Moreover, condition ME 2 fixes $\mathcal{B}$ as the commutant algebra.

However, since $\mathcal{A}$ often does not admit a trace, we cannot represent ω by a density operator, and therefore we cannot use the spectrum of the density operator to characterize ω. Surprisingly, it is equilibrium statistical mechanics, which provides the notion to generalize. In the finite dimensional context, we can consider every density operator as a canonical equilibrium state, and determine from it the Hamiltonian of the system. This in turn defines a time evolution. Note that the Hamiltonian is only defined up to a constant, so we cannot expect to reconstruct the eigenvalues of H, but only the spectrum of the Liouville operator $\sigma \mapsto i[\sigma, H]$, which generates the dynamics on density operators, and has eigenvalues $i(E_n - E_m)$, when the E_n are the eigenvalues of H. The connection between the time evolutions and equilibrium states makes sense also for von Neumann algebras, and can be seen as the physical interpretation of modular theory [13].

We begin the outline of this theory with the anti-linear operator S on $\mathcal{H}$ by

$$S(A\Omega) = A^*\Omega, \quad A \in \mathcal{A}. \tag{13}$$

It turns out to be closable, and we denote its closure by the same letter. As a closed operator S admits a polar decomposition

$$S = J\Delta^{1/2}, \tag{14}$$

which defines the anti-unitary *modular conjugation* J and the positive *modular operator* Δ.

Let us calculate Δ in the standard situation, where $\mathcal{H} = \mathcal{K} \otimes \mathcal{K}$, and $\mathcal{A} = \mathcal{B}(\mathcal{K}) \otimes \mathbb{1}$ respectively $\mathcal{B} = \mathbb{1} \otimes \mathcal{B}(\mathcal{K})$, and Ω is in Schmidt form (12). Due to assumption ME 7 (cyclicity), the orthonormal systems e_α and f_α have to be even complete (i.e., bases). Now consider (13) with $A = (|e_\beta\rangle\langle e_\gamma|) \otimes \mathbb{1}$, which becomes

$$S(c_\gamma e_\beta \otimes f_\gamma) = c_\beta e_\gamma \otimes f_\beta, \tag{15}$$

from which we readily get

$$\Delta^{1/2} = \rho^{1/2} \otimes \rho^{-1/2}, \quad \text{and} \quad J = F(\Theta \otimes \Theta), \tag{16}$$

where $\rho = \sum_\alpha c_\alpha^2 |e_\alpha\rangle\langle e_\alpha|$ is the reduced density operator, $F\phi_1 \otimes \phi_2 = \phi_2 \otimes \phi_1$ is the flip operator and Θ denotes complex conjugation in the e_n basis. The time evolution with Hamiltonian $H = -\log\rho + c\mathbb{1}$, for which ω is now the equilibrium state with unit temperature, is then given by $\mathcal{E}_t(A) \otimes \mathbb{1} = \Delta^{it}(A \otimes \mathbb{1})\Delta^{-it}$.

In the case of general von Neumann algebras, the spectrum of Δ need no longer be discrete, and it can be a general positive, but unbounded selfadjoint operator. It turns out that Δ^{it} still defines a time evolution on the algebra $\mathcal{A}$, the so-called *modular evolution* The equilibrium condition cannot be written directly in the Gibbs form $\rho \propto \exp(-H)$, since there is no density matrix any more, but has to be replaced by the so-called KMS-condition, a boundary condition for the analytic continuation of correlation functions [13, 27] which links the modular evolution to the state.

In the standard situation, the eigenvalue 1 of Δ plays a special role, because it points to degeneracies in the Schmidt spectrum. In the extreme case of a maximally entangled state all c_α are equal, and $\Delta = \mathbb{1}$ or, equivalently, S is anti-unitary. This characterization of maximal entanglement carries over to the von Neumann algebra case: S is anti-unitary if and only if for all $A_1, A_2 \in \mathcal{A}$

$$\langle\Omega, A_1A_2\Omega\rangle = \langle A_1^*\Omega, A_2\Omega\rangle = \langle SA_1\Omega, SA_2^*\Omega\rangle = \langle A_2^*\Omega, A_1\Omega\rangle = \langle\Omega, A_2A_1\Omega\rangle.$$

This is precisely the trace property ME 4.

5.4 Characterization by the EPR-doubles property

In the original EPR-argument it is crucial that certain observables of Alice and Bob are perfectly correlated, so that Alice can find the values of observables on Bob's side with certainty, without Bob having to carry out this measurement. An approach to studying such correlations was proposed recently by Arens and Varadarajan [28]. The basic idea, stripped of some measure theoretic overhead, and extended to the more general bipartite systems considered here [29], rests on the following definition. Let $\mathcal{A}, \mathcal{B}$ be commuting observable algebras and ω a state on an algebra containing both $\mathcal{A}$ and $\mathcal{B}$. Then we say that an element $B \in \mathcal{B}$ is an *EPR-double* of $A \in \mathcal{A}$, or that A and B are doubles (of each other) if

$$\omega\big((A^* - B^*)(A - B)\big) = \omega\big((A - B)(A^* - B^*)\big) = 0. \tag{17}$$

Of course, when A and B are hermitian, the two expressions coincide, and in this case there is a simple interpretation of equation (17). Since A and B commute, we can consider their joint distribution (measuring the joint spectral resolution of A and B). Then $(A-B)^2$ is a positive quantity, which has vanishing expectation if and only if the joint distribution is concentrated on the diagonal, i.e., if the measured values coincide with probability one.

Basic properties are summarized in the following Lemma.

Lemma 1 *Let ω be a state on a C^*-algebra containing commuting subalgebras $\mathcal{A}$ and $\mathcal{B}$. Then*

- *(i) A and B are doubles iff for all C in the ambient observable algebra we have $\omega(AC) = \omega(BC)$ and $\omega(CA) = \omega(CB)$.*

(ii) If A_1, A_2 have doubles B_1, B_2, then $A_1^*, A_1 + A_2$, and A_1A_2 have doubles $B_1^*, B_1 + B_2$, and B_2B_1, respectively.

(iii) When A and B are normal ($AA^* = A^*A$), and doubles of each other, then so are $f(A)$ and $f(B)$, where f is any continuous complex valued function on the spectrum of A and B, evaluated in the functional calculus.

(iv) When $\mathcal{A}$ and $\mathcal{B}$ are von Neumann algebras, and ω is a normal state, and observables A_n with doubles B_n converge in weak-topology to A, then every cluster point of the sequence B_n is a double of A.*

(v) Suppose that ω restricted to $\mathcal{B}$ is faithful (i.e., $\mathcal{B} \ni B \geq 0$ and $\omega(B) = 0$ imply $B = 0$). Then every $A \in \mathcal{A}$ admits at most one double.

Proof. (i) One direction is obvious by setting $C = A^* - B^*$. The other direction follows from the Schwartz inequality $|\omega(X^*Y)|^2 \leq \omega(X^*X)\omega(Y^*Y)$.

The remaining items follow directly from (i). (iii) is obvious from (ii) for polynomials in A and A^*, and extends to continuous functions by taking norm limits on the polynomial approximations to f provided by the Stone-Weierstraß approximation theorem. For (iv) one has to use the weak*-continuity of the product in each factor separately (see e.g. [30, Theorem 1.7.8]). □

In the situation we have assumed for modular theory, we can give a detailed characterization of the elements admitting a double:

Proposition 1 *Suppose $\mathcal{A}$ and $\mathcal{B} = \mathcal{A}'$ are von Neumann algebras on a Hilbert space $\mathcal{H}$, and the state ω is given by a vector $\Omega \in \mathcal{H}$, which is cyclic for both $\mathcal{A}$ and $\mathcal{B}$. Then for every $A \in \mathcal{A}$ the following conditions are equivalent:*

(i) A has an EPR-double $B \in \mathcal{B}$.

(ii) A is in the centralizer *of the restricted state, i.e., $\omega(AA_1) = \omega(A_1A)$ for all $A_1 \in \mathcal{A}$.*

(iii) A is invariant under the modular evolution $\Delta^{it}A\Delta^{-it} = A$ for all $t \in \mathbb{R}$.

In this case the double is given by $B = JA^*J$.

Proof. (i) ⇒ (ii) When A has a double B, we get $\omega(AA_1) = \omega(BA_1) = \omega(A_1B) = \omega(A_1A)$ for all A_1 in the ambient observable algebra.

(ii) ⇔ (iii) This is a standard result (see, e.g., [31, Prop. 15.1.7]).

(iii) ⇒ (i) Since $\Delta^{it}\Omega = \Omega$, (iii) implies $\Delta^{it}A\Omega = A\Omega$, so $A\Omega$ is an eigenvector for eigenvalue 1 of the unitary Δ^{it} and $\Delta A\Omega = A\Omega$. By the same token, $\Delta A^*\Omega = A^*\Omega$. We claim that in that case $B = JA^*J \in \mathcal{B}$ is a double of A in $\mathcal{B}$: We have $B\Omega = JA^*J\Omega = JA^*\Omega = JSA\Omega = \Delta A\Omega = A\Omega$ and, similarly, $B^*\Omega = A^*\Omega$. From this (i) follows immediately.

The formula for B was established in the last part of the proof. Uniqueness follows from Lemma 1. □

Two special cases are of interest. On the one hand, in the standard case of a pure bipartite state we get a complete characterization of the observables which posses a double: they are exactly the ones commuting with the reduced density operator [28]. On the other hand, we can ask under what circumstances *all* $A \in \mathcal{A}$ admit a double. Clearly, this is the case when the centralizer in (ii) of the Proposition is all of $\mathcal{A}$, i.e., if and only if the restricted state is a trace. Again this characterizes the everybody's maximally entangled states on finite dimensional algebras, and the unique infinite dimensional one for hyperfinite von Neumann algebras.

6 The original EPR state

In their famous 1935 paper [5] Einstein, Podolsky and Rosen studied two quantum particles with perfectly correlated momenta and perfectly anticorrelated positions. It is immediately clear that such a state does not exist in the standard framework of Hilbert space theory: the difference of the positions is a self-adjoint operator with purely absolutely continuous spectrum, so whatever density matrix we choose, the probability distribution of this quantity will have a probability density with respect to Lebesgue measure, and cannot be concentrated on a single point. Consequently, the wave function written in [5] is a pretty wild object. Essentially it is $\Psi(x_1, x_2) = c\delta(x_1 - x_2 + a)$, with the Dirac delta function, and c a "normalization factor" which must vanish, because the normalization integral for the delta function is undefined, but infinite if anything.

How could such a profound physical argument be based on such an ill-defined object? The answer is probably that the authors were completely aware that they were really talking about a limiting situation of more and more sharply peaked wave functions. We could model them by a sequence of more and more highly squeezed two mode Gaussian states (cf. Subsection 6.5), or some other sequence representation of the delta function. The key point is that the main argument does not depend on the particular approximating sequence. But then we should also be able to discuss the limiting situation directly in a rigorous way, and extract precisely what is common to all approximations of the EPR state.

6.1 Definition

In this section we consider a family of singular states, which describes quite well what Einstein Podolsky and Rosen may have had in mind. Throughout we assume we are in the usual Hilbert space $\mathcal{H} = \mathcal{L}^2(\mathbb{R}^2)$ for describing two canonical degrees of freedom, with position and momentum operators Q_1, Q_2, P_1, P_2. The basic observation is that the operators $P_1 + P_2$ and $Q_1 - Q_2$ commute as a consequence of the Heisenberg commutation relations. Therefore we can evaluate in the functional calculus (i.e., using a joint spectral resolution) any function of the form $g(P_1 + P_2, Q_1 - Q_2)$, where $g : \mathbb{R}^2 \to \mathbb{C}$ is an arbitrary bounded continuous function. We define an *EPR-state* as any state ω such that

$$\omega\Big(g(P_1 + P_2, Q_1 - Q_2)\Big) = g(0, a)\ , \tag{18}$$

where a is the fixed distance between the particles. Several comments are in order. First of all, if we take any sequence of vectors to "approximate" the EPR wave function (and adjust normalization on the way), weak*-cluster points of the corresponding sequence of pure states exist by compactness of the state space, and all these will be EPR states in the sense of our definition. Secondly, condition (18) does not fix ω uniquely. Indeed, different approximating sequences may lead to different ω. Even for a fixed approximating sequence it is rarely the case that the expectation values of *all* bounded operators converge, so the sequence will have many different cluster points. Thirdly, the existence of EPR states can also be seen more directly: the algebra of bounded continuous functions on $\mathbb{R}^2$ is faithfully represented in $\mathcal{B}(\mathcal{H})$ (i.e., $g(P_1 + P_2, Q_1 - Q_2) = 0$ only when g is the zero function). On that algebra the point evaluation at $(0, a)$ is a well defined state, so any Hahn-Banach extension of this state to all of $\mathcal{B}(\mathcal{H})$ will be an EPR state.[f]

In our further analysis we will only look at properties which are common to all EPR states, and which are hence independent of any choice of approximating sequences. The basic technique for extracting such properties from (18) is to use positivity of ω in the form of the Schwartz inequality $|\omega(A^*B)| \leq \omega(A^*A)\omega(B^*B)$. For example, we get

$$\omega\Big(X\widehat{g}\Big) = \omega\Big(\widehat{g}\, X\Big) = g(0,a)\omega(X)\ , \tag{19}$$

where $\widehat{g}$ is shorthand for $g(P_1 + P_2, Q_1 - Q_2)$ for some bounded continuous function g, and $X \in \mathcal{B}(\mathcal{H})$ is an arbitrary bounded operator. This is shown by taking $A = X^*$ and $B = (\widehat{g} - g(0,a)\mathbb{1})$ (or $A = (\widehat{g} - g(0,a)\mathbb{1})$ and $B = X$) in the Schwartz inequality.

6.2 Restriction to the CCR-algebra

Next we consider the expectations of *Weyl operators*

$$\begin{aligned} \mathbf{W}(\xi_1, \xi_2, \eta_1, \eta_2) &= e^{i(\xi_1 P_1 + \xi_2 P_2 - \eta_1 Q_1 - \eta_2 Q_2)} \\ &= e^{i(\vec{\xi}\cdot\vec{P} - \vec{\eta}\cdot\vec{Q})}\ . \end{aligned} \tag{20}$$

Obviously, if $\xi_1 = \xi_2$ and $\eta_1 = -\eta_2$, which we will abbreviate as $(\vec{\xi}, \vec{\eta}) \in S$, we have $\mathbf{W}(\vec{\xi}, \vec{\eta}) = \widehat{g}$ for a uniformly continuous g, so (18) determines the expectation. Combining it with Equation (19) we get:

$$\omega\Big(\mathbf{W}(\vec{\xi}, \vec{\eta})X\Big) = \omega\Big(X\mathbf{W}(\vec{\xi}, \vec{\eta})\Big) = \omega(X) \quad \text{for } (\vec{\xi}, \vec{\eta}) \in S\ . \tag{21}$$

In particular, the state is invariant under all phase space translations by vectors in S.

This is already sufficient to conclude that the state is purely singular, i.e., that $\omega(K) = 0$ for every compact operator, and in particular for all finite dimensional projections. An even stronger statement is that the restrictions to Alice's and Bob's subsystem are purely singular.

Lemma 2 *For any EPR state, and any compact operator K, $\omega(K \otimes \mathbb{1}) = 0$.*

Proof. Indeed the restricted state is invariant under *all* phase space translations, since we can extend $\mathbf{W}(\xi, \eta)$ to a Weyl operator of the total system, i.e., $\mathbf{W}'(\xi, \eta) = \mathbf{W}(\xi, \xi, \eta, -\eta) \cong \mathbf{W}(\xi, \eta) \otimes \mathbf{W}(\xi, -\eta)$, with $(\xi, \xi, \eta, -\eta) \in S$, and

$$\omega\big((\mathbf{W}(\xi, \eta)A\mathbf{W}(\xi, \eta)^*) \otimes \mathbb{1}\big) = \omega\big(\mathbf{W}'(\xi, \eta)(A \otimes \mathbb{1})\mathbf{W}'(\xi, \eta)^*)\big)\ . \tag{22}$$

Now consider a unit vector χ with bounded support in position space, and let $K = |\chi\rangle\langle\chi|$ be the corresponding one-dimensional projection. Then sufficiently widely space translates

[f]The reason for defining EPR-states with respect to *continuous* functions of $P_1 + P_2$ and $Q_1 - Q_2$ rather than, say, measurable functions, is that we need faithfulness. The functional calculus is well defined also for measurable functions, but some functions will evaluate to zero. In particular, for the function $g(p, x) = 1$ for $x = a$ and $p = 0$, but $g(p, x) = 0$ for all other points, we get $g(P_1 + P_2, Q_1 - Q_2) = 0$, because the joint spectrum of these operators is purely absolutely continuous. Hence condition (18), extended to measurable functions would require the expectation of the zero operator to be 1.

$\mathbf{W}(n\xi_0, 0)\chi$ are orthogonal, and hence, for all N, the operator

$$K_N = \sum_{n=1}^{N} \mathbf{W}(n\xi_0, 0) K \mathbf{W}(n\xi_0, 0^*)$$

is bounded by $\mathbb{1}$. Hence $N\omega(K) = \omega(K_N) \leq \omega(\mathbb{1}) = 1$, and $\omega(K) = 0$. Since vectors of compact support are norm dense in Hilbert space, the conclusion holds for arbitrary χ. Since finite rank operators are dense in the space of compact operators, the statement follows. □

For other Weyl operators we get the expectations from the Weyl commutation relations

$$\mathbf{W}(\vec{\xi}, \vec{\eta})\mathbf{W}(\vec{\xi}\,', \vec{\eta}\,') = \mathrm{e}^{\mathrm{i}\frac{\sigma}{2}} \; \mathbf{W}(\vec{\xi} + \vec{\xi}\,', \vec{\eta} + \vec{\eta}\,') \quad \text{with } \sigma = \vec{\xi} \cdot \vec{\eta}\,' - \vec{\xi}\,' \cdot \vec{\eta}\,. \tag{23}$$

This is just a form of the Heisenberg commutation relations. Now S is a so-called *maximal isotropic* subspace of phase space, which is to say that the commutation phase σ vanishes for $(\vec{\xi}, \vec{\eta}), (\vec{\xi}\,', \vec{\eta}\,') \in S$, and no subspace of phase space strictly including S has the same property.

For a point $(\vec{\xi}, \vec{\eta})$ in phase space, which does not belong to S, we can find some vector $(\vec{\xi}\,', \vec{\eta}\,') \in S$ such that the commutation phase $\mathrm{e}^{\mathrm{i}\sigma} \neq 1$ is non trivial. Combining the Weyl relations (23) with the invariance (21) gives

$$\omega(\mathbf{W}(\vec{\xi}, \vec{\eta})) = \omega(\mathbf{W}(\vec{\xi}\,', \vec{\eta}\,')\mathbf{W}(\vec{\xi}, \vec{\eta})) = \mathrm{e}^{\mathrm{i}\sigma}\omega(\mathbf{W}(\vec{\xi}, \vec{\eta})\mathbf{W}(\vec{\xi}\,', \vec{\eta}\,')) = \mathrm{e}^{\mathrm{i}\sigma}\omega(\mathbf{W}(\vec{\xi}, \vec{\eta}))$$

which implies that the expectation values

$$\omega\Big(\mathbf{W}(\vec{\xi}, \vec{\eta})\Big) = 0 \qquad \text{for } (\vec{\xi}, \vec{\eta}) \notin S \tag{24}$$

must vanish. With equations (21,24) we have a complete characterization of the state ω restricted to the "CCR-algebra", which is just the C*-algebra generated by the Weyl operators. Since this is a well-studied object, one might make these equations the starting point of an investigation of EPR states. However, one can see that (18) is strictly stronger: there are states which look like ω on the CCR-algebra, but which give an expectation in (18) corresponding to a limit of states going to infinity instead of going to zero.

6.3 EPR-correlations

How about the correlation property, which is so important in the EPR-argument? The best way to show this is the 'double' formalism of Section 5.4 in which we denote by $\mathcal{Z}$ the norm closed subalgebra of operators on $\mathcal{L}^2(\mathbb{R})$ generated by all operators of the form $f(\xi P + \eta Q)$, where $f : \mathbb{R} \to \mathbb{C}$ is an arbitrary *uniformly continuous* function evaluated in the functional calculus on a real linear combination $\xi P + \eta Q$ of position and momentum[g] This algebra is fairly large: it contains many observables of interest, in particular all Weyl operators and all compact operators. It is closed under phase space translations, and these act continuously[h] in the sense that, for $Z \in \mathcal{Z}$, $\|\mathbf{W}(\xi, \eta) Z \mathbf{W}(\xi, \eta)^* - Z\| \to 0$ as $(\xi, \eta) \to 0$.

Theorem 4 *All operators of the form $Z \otimes \mathbb{1}$ with $Z \in \mathcal{Z}$ have doubles in the sense of equation* (17). *Moreover, the double of $Z \otimes \mathbb{1}$ is $\mathbb{1} \otimes Z^T$, where Z^T denotes the transpose (adjoint followed by complex conjugation) in the position representation.*

[g]The same type of operators, although motivated by a different argument already appears in [8].

[h]This continuity is crucial in the correspondence theory set out in [15]. We where not able to prove the analogue of Theorem 4 by only assuming this continuity.

Proof. We only have to show that for $f(\xi P + \eta Q) \otimes \mathbb{1} = f(\xi P_1 + \eta Q_1)$ we get the double $\mathbb{1} \otimes f(-\xi P + \eta Q) = f(-\xi P_2 + \eta Q_2)$, when f, ξ, and η are as in the definition of $\mathcal{Z}$. By the general properties of the double construction this will then automatically extend to operator products and norm limits.

Fix $\varepsilon > 0$. Since f is uniformly continuous, there is some $\delta > 0$ such that $|f(x) - f(y)| \leq \varepsilon$ whenever $|x - y| \leq \delta$. Now pick a continuous function $h : \mathbb{R} \to [0,1] \subset \mathbb{R}$ such that $h(0) = 1$, $h(t) = 0$ for $|t| > \delta$. We consider the operator

$$\begin{aligned} M &= \big(f(\xi P_1 + \eta Q_1) - f(-\xi P_2 + \eta Q_2)\big) h\big(\xi(P_1 + P_2) + \eta(Q_1 - Q_2)\big) \\ &= F(\xi P_1 + \eta Q_1, -\xi P_2 + \eta Q_2) \end{aligned}$$

where $F(x,y) = (f(x) - f(y))h(x-y)$ and this function is evaluated in the functional calculus of the commuting selfadjoint operators $(\xi P_1 + \eta Q_1)$ and $(-\xi P_2 + \eta Q_2)$. But the real valued function F satisfies $|F(x,y)| \leq \varepsilon$ for all (x,y): when $|x - y| > \delta$ the h-factor vanishes, and on the strip $|x - y| \leq \delta$ we have $|f(x) - f(y)| \leq \varepsilon$. Therefore $\|M\| \leq \varepsilon$. Let X be an arbitrary operator. Then

$$\big|\omega\big(\big[f(\xi P_1 + \eta Q_1) - f(-\xi P_2 + \eta Q_2)\big] X\big)\big| = |\omega\big(MX\big)| \leq \|M\| \, \|X\| \leq \varepsilon \|X\| \,.$$

Here we have added a factor $h(\xi(P_1 + P_2) + \eta(Q_1 - Q_2))$ at the second equality sign, which we may because of (19), and because h is a function of the appropriate operators, which is $= 1$ at the origin. Since this estimate holds for any ε, we conclude that the first relation in Lemma 1.1 holds. The argument for the second relation is completely analogous. □

6.4 Infinite one-shot entanglement

In order to show that the EPR state is indeed highly entangled, let us verify that it contains infinite one-shot entanglement in the sense forbidden by Theorem 1. The local operations needed to extract a d-dimensional system will be simply the restriction to a subalgebra. In other words, we will construct subalgebras $\mathcal{A}_d \subset \mathcal{A}$ and $\mathcal{B}_d \subset \mathcal{B}$ such that the state ω restricted to $\mathcal{A}_d \otimes \mathcal{B}_d$ will be a maximally entangled pure state of d-dimensional systems.

The matrix algebras $\mathcal{A}_d, \mathcal{B}_d$ are best seen to be generated by Weyl operators, satisfying a discrete version of the canonical commutation relations (23), with the addition operation on the right hand side replaced by the addition in a finite group. Let $\mathbb{Z}_d$ denote the cyclic group of integers modulo d. With the canonical basis $|k, \ell\rangle,\ k, \ell \in \mathbb{Z}_d$ we introduce the Weyl operators

$$\mathbf{w}(n_1, m_1, n_2, m_2)|k, \ell\rangle = \zeta^{n_1(k - m_1) + n_2(\ell - m_2)}|k - m_1, \ell - m_2\rangle, \tag{25}$$

where $\zeta = \exp(2\pi i/d)$ is the dth root of unity. These are a basis of the vector space $\mathcal{B}(\mathbb{C}^d \otimes \mathbb{C}^d)$, which shows that this algebra is generated by the four unitaries $u_1 = \mathbf{w}(1,0,0,0), v_1 = \mathbf{w}(0,1,0,0), u_2 = \mathbf{w}(0,0,1,0)$ and $v_2 = \mathbf{w}(0,0,0,1)$. They are defined algebraically by the relations $v_k u_k = \zeta u_k v_k, k = 1,2$, and $u_1^d = u_2^d = v_1^d = v_2^d = \mathbb{1}$. The one dimensional projection onto the standard maximally entangled vector $\Omega = d^{-1/2} \sum_k |kk\rangle$ can be expressed in the basis (25) as

$$
\begin{aligned}
|\Omega\rangle\langle\Omega| &= \frac{1}{d^2}\sum_{n,m}\mathbf{w}(n,m,-n,m) \\
&= \frac{1}{d^2}\sum_{n,m}(u_1u_2^{-1})^n(v_1v_2)^m,
\end{aligned} \tag{26}
$$

which will be useful for computing fidelity.

In order to define the subalgebras extracting the desired entanglement we first define operators U_1, V_1 in Alice's subalgebra and U_2, V_2 in Bob's, which satisfy the above relations and hence generate two copies of the $d \times d$ matrices. It is easy to satisfy the commutation relations $V_kU_k = \zeta U_kV_k$, by taking appropriate Weyl operators, say

$$\widetilde{U}_1 = \mathrm{e}^{iQ_1},\ \widetilde{U}_2 = \mathrm{e}^{i(Q_2-a)}, \quad \text{and} \quad \widetilde{V}_k = \mathrm{e}^{i\xi P_k} \tag{27}$$

with $\xi = 2\pi/d$. The tilde indicates that these are not quite the operators yet we are looking for, because they do not satisfy the periodicity relations: $\widetilde{U}_1^d = \exp(idQ_1) \neq \mathbf{1}$, and similarly for U_2^d and $\widetilde{V}_k$. We will denote by $\widetilde{\mathcal{A}}$ the C*-algebra, generated by the operators $\widetilde{U}_1, \widetilde{V}_1$ (27). The algebra $\widetilde{\mathcal{B}}$ is constructed analogously. Then by virtue of the commutation relations $\widetilde{U}_1^d$ and $\widetilde{V}_1^d$ commute with all other elements of $\widetilde{\mathcal{A}}$, i.e., they belong to the *center* $\mathcal{C}_A \subset \widetilde{\mathcal{A}}$, which represents the classical variables of the system. In the same manner, $\widetilde{U}_2^d$ and $\widetilde{V}_2^d$ generate the center $\mathcal{C}_B$ of Bob's algebra[i] $\widetilde{\mathcal{B}}$.

If we take any continuous function (in the functional calculus) of a hermitian or unitary element of $\mathcal{C}_A$, it will still be in $\mathcal{C}_A$. If we take a measurable (possibly discontinuous) function the result may fail to be in $\mathcal{C}_A$, but it still commutes with all elements of $\widetilde{\mathcal{A}}$ (and analogously for Bob's algebras). In particular, we construct the operators

$$\widehat{U}_k = \left(\widetilde{U}_k^d\right)^{1/d}, \tag{28}$$

where the dth root of numbers on the unit circle is taken with a branch cut on the negative real axis. This branch cut makes the function discontinuous, and also makes this odd-looking combination very different from $\widetilde{U}_k$. We now define $\widehat{V}_k$ analogously, and set

$$U_k = \widehat{U}_k^{-1}\widetilde{U}_k \quad \text{and} \quad V_k = \widehat{V}_k^{-1}\widetilde{V}_k \tag{29}$$

for $k = 1, 2$. Then since $\widehat{U}_k, \widehat{V}_k$ commute with $\widetilde{\mathcal{A}}\otimes\widetilde{\mathcal{B}}$, the commutation relations $V_kU_k = \zeta U_kV_k$ still hold, but in addition we have $U_k^d = \mathbf{1}$, because $\widehat{U}_k^d = \widetilde{U}_k^d$. It remains to show that on the finite dimensional algebras generated by these operators, the given state is a maximally entangled pure state. We will verify this by computing the fidelity, i.e., the expectation of

[i] The C*-algebra $\widetilde{\mathcal{A}}$ is isomorphic to the continuous sections in an C*-algebra bundle over the torus, where each fiber is a copy of the algebra $\mathcal{A}_d$. Such a bundle is called *trivial*, if it is isomorphic to the tensor product $\mathcal{A}_d \otimes \mathcal{C}_A$. This would directly give us the desired subalgebra $\mathcal{A}_d$ as a subalgebra of $\mathcal{A}$. However, this is bundle is not trivial [32, 33]. In order to "trivialize" the bundle, we are therefore forced to go beyond norm continuous operations, which respect the continuity of bundle sections. Instead we have to go to the measurable functional calculus, and introduce an operation on the fibers, which depends discontionuously on the base point, through the introduction of a branch cut.

the projection (26):

$$\omega\left(\frac{1}{d^2}\sum_{n,m}(U_1U_2^{-1})^n(V_1V_2)^m\right)=1. \tag{30}$$

Proof of this equation. We have shown in Section 6.3 that $\widetilde{U}_1$ and $\widetilde{U}_2$ are EPR-doubles. This property transfers to arbitrary continuous functions of $\widetilde{U}_1$ and $\widetilde{U}_2$ by Lemma 1 and uniform approximation of continuous functions by polynomials. However, because the state ω is not normal, it does *not* transfer automatically to the measurable functional calculus and hence not automatically to $\widehat{U}_1$ and $\widehat{U}_2$. We claim that this is true nonetheless.

Denote by $r_d(z) = z^{1/d}$ the dth root function with the branch cut as described, and let f_ϵ be a continuous function from the unit circle to the unit interval $[0,1]$ such that $f_\epsilon(z)=1$ except for z in an ϵ-neighborhood of $z=-1$ in arclength, and such that $f_\epsilon(-1)=0$. Then the function $z\mapsto f_\epsilon(z)r_d(z)$ is continuous. Then, since $\widetilde{U}_1^d$ and $\widetilde{U}_2^d$ are doubles, so are $f_\epsilon(\widetilde{U}_1^d)$, $f_\epsilon(\widetilde{U}_1^d)\widehat{U}_1$ and their counterparts. Note that both of these commute with all other operators involved. Hence (using the notation $|X|^2 = X^*X$ or $|X|^2 = XX^*$, which coincide in this case)

$$\omega\Big(f_\epsilon(\widetilde{U}_1^d)^2|\widehat{U}_1-\widehat{U}_2|^2\Big)=\omega\Big(\big|f_\epsilon(\widetilde{U}_1^d)\widehat{U}_1-f_\epsilon(\widetilde{U}_2^d)\widehat{U}_2\big|\Big)=0, \tag{31}$$

where the first equality holds by expanding the modulus square, and applying the double property of $f_\epsilon(\widetilde{U}_1^d)$ where appropriate. On the other hand, we have

$$\omega\Big(\big(\mathbb{1}-f_\epsilon(\widetilde{U}_1^d)^2\big)|\widehat{U}_1-\widehat{U}_2|^2\Big)\leq 4\omega\big(\mathbb{1}-f_\epsilon(\widetilde{U}_1^d)^2\big)\leq 4\frac{\epsilon}{\pi}, \tag{32}$$

because $\|\widehat{U}_1-\widehat{U}_2\|\leq 2$, and $0\leq f_\epsilon(\widetilde{U}_1^d)\leq \mathbb{1}$. For the estimate we used that $f_\epsilon(z)^2$ for all z on the unit circle except a section of relative size $2\epsilon/(2\pi)$, and that the probability distribution for the spectrum of $\widetilde{U}_1^d$ is uniform, because the expectation of all powers $(\widetilde{U}_1^d)=\exp(indQ_1)$ vanishes.

Adding (31) and (32) we find that $\omega\big(|\widehat{U}_1-\widehat{U}_2|^2\big)\leq 4\epsilon/\pi$ for every ϵ, and hence that $\widehat{U}_1$ and $\widehat{U}_2$ are EPR doubles as claimed. The proof that $\widehat{V}_1$ and $\widehat{V}_2^*$ are likewise doubles (just as $\widetilde{V}_1$ and $\widetilde{V}_2^*$) is entirely analogous. Hence U_1 and U_2 as well as V_1 and V_2 are also doubles. Applying this property in the fidelity expression (30) we find that every term has expectation one, so that with the prefactor d^{-2} the d^2 terms add up to one as claimed. □

6.5 EPR states based on two mode Gaussians

In this section we will deviate from the announcement that we intended to study only such properties of EPR states which follow from the definition alone, and are hence common to all EPR states. The reason is that there is one particular family, which has a lot of additional symmetry, and hence more operators admitting doubles, than general EPR states. Moreover, it is very well known. In fact, most people working in quantum optics probably have a very concrete picture of the EPR state, or rather of an approximation to this state: since Gaussian states play a prominent role in the description of lasers, it is natural to consider a Gaussian wave function of the form

$$\Psi_\lambda(q_1, q_2) = \frac{1}{\sqrt{\pi}} \exp\left(-\frac{1-\lambda}{4(1+\lambda)}(q_1 - q_2)^2 - \frac{1+\lambda}{4(1-\lambda)}(q_1 + q_2)^2 \right) \tag{33}$$

$$\Psi_\lambda = \sqrt{1-\lambda^2} \sum_{n=0}^{\infty} \lambda^n \, \mathbf{e}_n \otimes \mathbf{e}_n \ , \tag{34}$$

where $\mathbf{e}_n$ denotes the eigenbasis of the harmonic oscillators $H_i = (P_i^2 + Q_i^2)/2$ $(i = 1, 2)$. This state is also known as the NOPA state ("squeezed vacuum"[34]), and the parameter $\lambda \in [0, 1)$ is related to the so-called squeezing parameter r by $\lambda = \tanh(r)$. Values around $r = 5$ are considered a good experimental achievement [35]. Of course, we are interested in the limit $r \to \infty$, or $\lambda \to 1$.

The λ-dependence of the wave function can also be written as

$$\Psi_\lambda(q_1, q_2) = \Psi_0(q_1 \mathrm{ch}\eta + \mathrm{q}_2 \mathrm{sh}\eta, -\mathrm{q}_1 \mathrm{sh}\eta + \mathrm{q}_2 \mathrm{ch}\eta) \ , \tag{35}$$

where the hyperbolic angle η is $r/2$. It is easy to see that for *any* wave function Ψ_0 the probability distributions of both $Q_1 - Q_2$ and $P_1 + P_2$ scale to a point measures at zero. Hence any cluster point of the associated sequence of states $\omega_\lambda(X) = \langle \Psi_\lambda, A\Psi_\lambda \rangle$ is an EPR state in the sense of our definition (with shift parameter $a = 0$). Note, however, that the family itself does not converge to any state: it is easy to construct observables X for which the expectation $\omega_\lambda(X)$ remains oscillating between 0 and 1 as $\lambda \to 1$. Here, as in the general case, a single state can only be obtained by going to a finest subsequence (or by taking the limit along an ultrafilter).

The virtue of the particular family (34) is that it has especially high symmetry: it is immediately clear that

$$\Big((f(H_1) - f(H_2) \Big) \Psi_\lambda = 0 \tag{36}$$

for all λ, and for all bounded functions $f : \mathbb{N} \to \mathbb{C}$ of the oscillator Hamiltonians H_1, H_2. This implies that $f(H_1)$ and $f(H_2)$ are doubles with respect to the state ω_λ for each λ. Clearly, this property remains valid in the limit along any subsequence, so all EPR-states obtained as cluster points of the sequence ω_λ also have $f(H_1)$ in their algebra of doubles. Consequently, the unitaries $U_k(t) = \exp(itH_k)$ are also doubles of each other, and the limiting states are invariant under the time evolution $U_{12}(t) = U_1(t) \otimes U_2(-t)$. This is certainly suggestive, because oscillator time evolutions have an interpretation as linear symplectic transformations on phase space: $Q_k \mapsto Q_k \cos t \pm P_k \sin t$ and $P_k \mapsto \mp Q_k \sin t + P_k \cos t$, where the upper sign holds for $k = 1$ and the lower for $k = 2$. The subspace S from Section 6.3 is invariant under such rotations, and one readily verifies that the time evolution $U_{12}(t)$ takes EPR states into EPR states. This certainly implies that by averaging we can generate EPR states invariant under this evolution, and we have clearly just constructed a family with this invariance.

As $\lambda \to 1$, the Schmidt spectrum in (34) becomes "flatter", which suggests that exchanging some labels n should also define a unitary with double. Let $p : \mathbb{N} \to \mathbb{N}$ denote an injective (i.e., one-to-one but not necessarily onto map). Then we define an isometry V_p by

$$V_p \, \mathbf{e}_n = \mathbf{e}_{p(n)} \tag{37}$$

with adjoint

$$V_p^* \, \mathbf{e}_n = \begin{cases} \mathbf{e}_{p^{-1}(n)} & \text{if } n \in p(\mathbb{N}) \\ 0 & \text{if } n \notin p(\mathbb{N}) \end{cases} \tag{38}$$

Let us assume that p has *finite distance*, i.e., there is a constant ℓ such that $|p(n)-n| \leq \ell$ for all $n \in \mathbb{N}$. We claim that in this case $V_p \otimes \mathbb{1}$ and $\mathbb{1} \otimes V_p^*$ are doubles in all EPR states constructed from the sequence (34). We show this by verifying that the condition holds approximately already for finite λ. Consider the vector

$$\begin{aligned} \Delta_\lambda &= \big(V_p \otimes \mathbb{1} - \mathbb{1} \otimes V_p^*\big)\Psi_\lambda \\ &= \sqrt{1-\lambda^2} \sum_{n=0}^{\infty} (\lambda^n - \lambda^{p(n)})\, \mathbf{e}_{p(n)} \otimes \mathbf{e}_n \ , \end{aligned} \tag{39}$$

where in the second summand we changed the summation index from n to $p(n)$, automatically omitting all terms annihilated by V_p^* according to (38). Since this is a sum of orthogonal vectors, we can readily estimate the norm by writing $(\lambda^n - \lambda^{p(n)}) = \lambda^n(1-\lambda^{p(n)-n})$:

$$\|\Delta_\lambda\|^2 \leq \max_n |1-\lambda^{p(n)-n}|^2 \leq |1-\lambda^{-\ell}|^2 \ , \tag{40}$$

which goes to zero as $\lambda \to 1$. Therefore

$$\omega_\lambda\Big(X\big(V_p \otimes \mathbb{1} - \mathbb{1} \otimes V_p^*\big)\Big) = \langle \Psi_\lambda, X\Delta_\lambda \rangle \to 0 \tag{41}$$

as $\lambda \to 1$. Hence $V_p \otimes \mathbb{1}$ and $\mathbb{1} \otimes V_p^*$ are doubles in any state defined by a limit of ω_λ along a subsequence, as claimed.

V_p is an isometry but not necessarily unitary. But it is effectively unitary under an EPR state: Since V_p is in the centralizer, we must have $\omega\big((\mathbb{1}-V_pV_p^*)\otimes\mathbb{1}\big) = \omega\big((\mathbb{1}-V_p^*V_p)\otimes\mathbb{1}\big) = 0$, although this operator is non-zero. This is in keeping with the general properties of EPR states, whose restrictions must be purely singular. In fact, $(\mathbb{1} - V_pV_p^*)$ is the projection onto those eigenstates $\mathbf{e}_n$ for which $n \notin p(\mathbb{N})$, and this set is finite: it has at most ℓ elements.[j]

It is interesting to note what happens if one tries to relax the finite distance condition. An extreme case would be the two isometries $V_{\text{even}}\,\mathbf{e}_n = \mathbf{e}_{2n}$ and $V_{\text{odd}}\,\mathbf{e}_n = \mathbf{e}_{2n+1}$. These cannot have doubles in *any* state, because the restriction ω_A of the state to the first factor would then have to satisfy $1 = \omega_A(V_{\text{even}}V_{\text{even}}^* + V_{\text{odd}}V_{\text{odd}}^*) = \omega_A(V_{\text{even}}^*V_{\text{even}} + V_{\text{odd}}^*V_{\text{odd}}) = \omega_A(\mathbb{1}+\mathbb{1}) = 2$. On the other hand, the norm of Δ_λ no longer goes to zero, and we get $\|\Delta_\lambda\|^2 \to 1/6$ instead.

To get infinite one-shot entanglement is easier than in the case of general EPR states: we can simply combine d periodic multiplication operators with d-periodic permutation operators to construct a finite Weyl-system of doubles.[k] In fact there is a very quick way to get high fidelity entangled pure states even for $\lambda < 1$ (see [37] for an application to Bell inequality violations). Consider the unitary operator $U_d : \mathcal{H} \to \mathcal{H} \otimes \mathbb{C}^d$ given by

$$U_d\, \mathbf{e}_{dk+r} = (\mathbf{e}_k \otimes \mathbf{e}_r^{(d)}) \ , \tag{42}$$

[j] For any $N > \ell$, consider the set $\{1, \ldots, N\}$. This has to contain at least the images of $\{1, \ldots, N-\ell\}$, hence it can contain at most ℓ elements not in $p(\mathbb{N})$.

[k] This is probably what the authors of [36] are trying to say.

for $k = 0, 1, \dots$ and $r = 0, 1, \dots, d-1$. Then

$$(U_d \otimes U_d)\Psi_\lambda = \Psi_{\lambda^d} \otimes \Psi_\lambda^{(d)} \tag{43}$$

with a λ-dependent normalized vector $\Psi_\lambda^{(d)} \in \mathbb{C}^d \otimes \mathbb{C}^d$ proportional to

$$\Psi_\lambda^{(d)} \propto \sum_{r=1}^{d} \lambda^r \, \mathbf{e}_r^{(d)} \otimes \mathbf{e}_r^{(d)} \, . \tag{44}$$

Note that the infinite dimensional factor on the right hand side of (43) is again a state of the form (34), however, a less entangled one with parameter $\lambda' = \lambda^d < \lambda$. The second factor, i.e., (43) becomes maximally entangled in the limit $\lambda \to 1$. Therefore the unitary $(U_d \otimes U_d)$ splits both Alice's and Bob's subsystem, so that the total system is split exactly into a less entangled version of itself and a pure, nearly maximally entangled d-dimensional pair. The local operation extracting entanglement from this state is to discard the infinite dimensional parts. Seen in one of the limit states of the family ω_λ this is maximally entangled, so equation (2) is satisfied with $\epsilon = 0$. Moreover, since the remaining system is of exactly the same type, the process can be repeated arbitrarily often.

6.6 Counterintuitive properties of the restricted states

Basically, subsection 6.3 shows that the EPR states constructed here do satisfy the requirements of the EPR argument. However, Einstein, Podolsky and Rosen do not consider the measurement of suitable periodic functions of Q_k or P_k but measurements of these quantities themselves [5]: What do EPR states have to say about these?

Unfortunately, the "values of momentum" found by Alice or Bob are not quite what we usually mean by "values": they are infinite with probability 1. To see this, recall the remark after eq. (21) that EPR states are invariant with respect to phase space translations with $\mathbf{W}(\vec{\xi}, \vec{\eta})$ with $(\vec{\xi}, \vec{\eta}) \in S$. Hence

$$\begin{aligned} &\omega\big(\mathbf{W}(\xi_1, 0, \eta_1, 0)(A \otimes \mathbb{1})\mathbf{W}(\xi_1, 0, \eta_1, 0)^*\big) \\ &\qquad = \omega\big(\mathbf{W}(\xi_1, \xi_1, \eta_1, -\eta_1)(A \otimes \mathbb{1})\mathbf{W}(\xi_1, \xi_1, \eta_1, -\eta_1)^*\big) = \omega(A \otimes \mathbb{1}). \end{aligned} \tag{45}$$

That is, the reduced state is invariant under all phase space translations. Now suppose that for some continuous function f with compact support we have $\omega(f(Q_1)) = \epsilon \neq 0$. Then we could add many (say N) sufficiently widely spaced translates of f to get an operator $F = \sum_i^N f(Q_1 + x_i \mathbb{1})$ with $\|F\| \leq \|f\|$ and $|N\epsilon| = |\omega(F)| \leq \|f\|$, which implies $\epsilon = 0$. Hence for every function with compact support we must have $\omega(f(Q_1)) = 0$. Note that this is possible only for singular states, since we can easily construct a sequence of compactly supported function increasing to the identity, whose ω expectations are all zero, hence fail to converge to 1.

In spite of being infinite, the "measured values" of Alice and Bob are perfectly correlated, which means that we have to distinguish different kinds of infinity. Such "kinds of infinity" are the subject of the topological theory of *compactifications* [38, 15]. The basic idea is very simple: consider some C*-algebra of bounded functions on the real line. Then the evaluations of the functions at a point, i.e., the functionals $x \mapsto f(x)$, are pure states on such an algebra, but compactness of the state space together with the Kreĭn-Milman Theorem [39] dictates that there are many more pure states. These additional pure states are interpreted as the points at infinity associated with the given observable algebra. The set of all pure states is called the Gel'fand spectrum of the commutative C*-algebra [13, Sec.2.3.5], and the algebra is known to be isomorphic to the algebra of continuous functions on this compact space. For the algebra of all bounded function the additional pure states are called free ultrafilters, for the algebra of all continuous bounded functions we get the points of the Stone-Čech-compactification, and for the algebra of uniformly continuous functions we get a still coarser notion of points at infinity. According to Section 6.3 these are the measured values, which will be perfectly correlated between Alice's and Bob's positions or momenta. It is not possible to exhibit any such value, because proving their mere existence already requires an argument based on the Axiom of Choice.

So do we have to be content with the statement that the measured values lie "out there on the infinite ranges, where the free ultrafilters roam?" Section 6.4 shows that for many concrete problems, involving not too large observable algebras, we can use the perfect correlation property quite well. A smaller algebra of observables means that many points of Gel'fand spectrum become identified, and some of these coarser points may have a direct physical interpretation. So the moral is not so much that compactification points at infinity are wild, pathological objects, but that they describe the way a sequence can go to infinity in the finest possible detail, which is just much finer that we usually want to know. The EPR correlation property holds even for such wild "measured values".

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