ENTANGLEMENT TRANSFORMATIONS OF PURE GAUSSIAN STATES

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We present a theory of entanglement transformations of Gaussian pure states with local Gaussian operations and classical communication. This is the experimentally accessible set of operations that can be realized with optical elements such as beam splitters, phase shifts and squeezers, together with homodyne measurements. We provide a simple necessary and sufficient condition for the possibility to transform a pure bipartite Gaussian state into another one. We contrast our criterion with what is possible if general local operations are available.

1 Introduction

For optical systems, both reliable sources producing Gaussian quantum states and efficient detection schemes such as homodyne detection are experimentally readily available [1]. Quantum states can also be manipulated in an accurate manner by means of optical elements such as beam splitters and phase plates. In fact, it has been realized that such systems with canonical variables – often referred to as continuous-variable systems – in Gaussian quantum states offer a promising potential for realistic quantum information processing. Teleportation schemes for continuous-variables have been theoretically proposed and experimentally implemented [2], generation of entanglement has been studied [3], and cloning [4], Bell-type schemes [5] and cryptographic protocols [6] have been suggested, to name a few. In addition to the work on purely optical systems, continuous atomic variables have been investigated in great detail, e.g., when studying collective spin states of a macroscopic sample of atoms [7]. From the perspective of the theory of quantum entanglement, the concepts of separability [8] and distillability [9], as well as entanglement quantification [10] have been extended to systems with canonical variables. All these investigations complement the original studies of entanglement in quantum information science in the finite-dimensional regime.

In the light of these successes it would be desirable to have tools at hand that help with deciding whether an envisioned task can be achieved in a feasible manner or not. One would then ask for mathematical criteria whether a certain state transformation can be performed
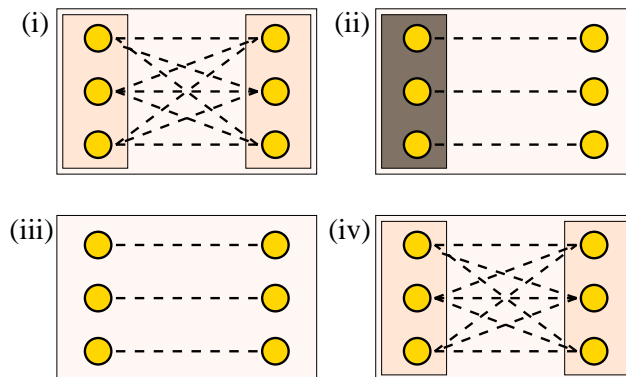
Fig. 1. Any transformation from one pure Gaussian state of $n \times n$ modes to another such state by means of local Gaussian operations with classical communication can be decomposed into three steps: (i) First, by means of local unitary Gaussian operations the state can be transformed into a product of n two-mode squeezed vacua as depicted in (ii). (iii) By means of (measuring) local Gaussian operations, classical communication of the outcomes and appropriate local Gaussian unitary operations one obtains a different product of two-mode squeezed vacua. (iv) The last step is to apply local Gaussian unitary operations in order to obtain the desired final state.

under a class of quantum operations that reflects the natural physical constraints of a given set-up. Such a tool proved very useful in the finite-dimensional setting. Often referred to as majorization criterion [11], it is a criterion for the set of local operations with classical communication (LOCC), which is the natural choice for finite-dimensional bi-partite systems: A pure state of a bi-partite system can be transformed into another pure state under LOCC if and only if the reductions of the initial state are more mixed than those of the final state. The term 'more mixed' has to be understood in the sense of majorization theory. Hence, the problem of deciding whether a particular transformation can be performed in principle – which can be an extremely difficult task – can be linked to a simple majorization relation.

A first step towards a theory of entanglement transformations for bi-partite systems with canonical variables has been undertaken in Ref. [12]: This criterion can in fact be applied to mixed Gaussian states of two modes, and the set of operations is a practically important set of feasible Gaussian operations. It does not, however, include measurements and classical communication. For pure states, in contrast, one could hope for a general criterion of entanglement transformation under all Gaussian local operations with classical communication (GLOCC). Gaussian operations are those operations that preserve the Gaussian character of states, and correspond exactly to those operations that can be implemented by means of optical elements such as beam splitters, phase shifts and squeezers together with homodyne measurements – all operations that are experimentally accessible with present technology [13, 14, 15]. A complete characterization of Gaussian operations has been presented in Refs. [13, 15].

In this paper we show that such a general criterion for pure-state Gaussian entanglement transformations can in fact be formulated: We present a necessary and sufficient criterion for pure-state entanglement transformations of bi-partite $n \times n$-mode systems under GLOCC. The criterion itself will turn out to be very simple: after having shown that each pure Gaussian

state of an $n \times n$ mode system is equivalent up to local Gaussian unitary operations to two-mode squeezed states (see also Ref. [17]), it will turn out that the criterion merely amounts to an element-wise comparison of squeezing vectors.

2   Notation

We will consider general Gaussian states ρ of $n \times n$ field modes, associated with the $4n$ canonical coordinates $R := (X_1, P_1, \ldots, X_{2n}, P_{2n})$. A *Gaussian state* is a state ρ the *characteristic function*

$$\chi_\rho(\xi) = \mathrm{tr}[\rho W_\xi] \tag{1}$$

of which is a Gaussian function in phase space, where $W_\xi := \exp(-i\xi^T R)$ is the *Weyl (displacement) operator*. The state ρ can then be written as

$$\rho = \pi^{-2n} \int_{\mathrm{R}^{4n}} d\xi \, \exp(-\frac{1}{4}\xi^T \gamma \xi + id^T \xi) W_\xi, \tag{2}$$

where $\gamma = \gamma^T$ is the *covariance matrix* (CM) incorporating the second moments, and d is the vector of the first moments. The first moments can be made to vanish via local operations and contain no information about entanglement. Only the second moments will therefore be of interest in the subsequent analysis. For example, the CM of a *two-mode squeezed state* ρ_r is given by

$$\gamma_r = \left(\begin{array}{cc} A_r & C_r \\ C_r & A_r \end{array} \right), \tag{3}$$

$A_r := \cosh(r)1$ and $C_r := \sinh(r)\Lambda$, with $\Lambda = \mathrm{diag}(1, -1)$, where $r \in [0, \infty)$ is the (two-mode) *squeezing parameter*. The canonical commutation relations can be formulated as $[R_j, R_k] = i\sigma_{j,k}$, $j, k = 1, \ldots, 4n$, with

$$\sigma = \bigoplus_{i=1}^{2n} \sigma_1, \ \ \sigma_1 = \left(\begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right) \tag{4}$$

being the *symplectic matrix*. In standard matrix theory the unitary diagonalization of a matrix and the resulting eigenvalues are key concepts. In the study of Gaussian states and their covariance matrices this role is taken by the idea of symplectic diagonalization. Any covariance matrix γ can be transformed into a diagonal matrix $S\gamma S^T$ by some symplectic transformation S. The diagonal elements of $S\gamma S^T$ form the symplectic spectrum. The *symplectic spectrum* of γ can be directly calculated as the modulus of the eigenvalues of $\sigma\gamma$. The *Heisenberg uncertainty principle* can be written as $\gamma \geq i\sigma$ [18], which is also necessary and sufficient for the real, positive matrix γ to be a CM.

   Gaussian operations [12, 13, 14, 15, 19] are those quantum operations (completely positive maps) that map all Gaussian states on Gaussian states. The practically most important subset is the set of Gaussian unitary operations, which are reflected by so-called *symplectic transformations*

$$\gamma \longmapsto S\gamma S^T \tag{5}$$

with $S\sigma S^T = \sigma$ on the level of covariance matrices. These are those unitary operations that can be realized by means of beam splitters, squeezers, and phase shifts. The most general "pure" Gaussian operation can be conceived as a concatenation of Gaussian unitary operations (on possibly a larger set of modes), together with homodyne measurements [13, 14, 15]. In the present context, it is most convenient to employ the isomorphism between completely positive maps and states [20], as explicitly analyzed and completely characterized in Ref. [13]. A general Gaussian map gives rise to a transformation $\gamma \longmapsto \gamma'$, where γ' is a Schur complement [21] of the form

$$\gamma' = \tilde{\Gamma}_1 - \tilde{\Gamma}_{12}(\tilde{\Gamma}_2 + \gamma)^{-1}\tilde{\Gamma}_{12}^T. \tag{6}$$

The CM

$$\Gamma = \left(\begin{array}{cc} \Gamma_1 & \Gamma_{12} \\ \Gamma_{12}^T & \Gamma_2 \end{array} \right) \tag{7}$$

defined on $4n$ modes specifies the actual Gaussian operation, where

$$\tilde{\Gamma} := (1 \oplus \Lambda)\Gamma(1 \oplus \Lambda), \tag{8}$$

and $\Lambda := \mathrm{diag}(1, -1, 1, -1, \dots, 1, -1)$.

3   A general criterion

Before we are in the position to state the theorem, we introduce a particularly useful normal form of Gaussian pure states of $n \times n$ modes. [a] This normal form has also been found independently by Holevo and Werner [16] and Botero and Reznik [17]. For reasons of completeness of this paper, we will however present an alternative proof in the notation used here in Appendix A.

Lemma 1 (Standard form of pure $n \times n$-states) *Any pure Gaussian state ρ of $n \times n$ modes can be transformed by local unitary Gaussian operations into a state which is a tensor product of n pure two-mode squeezed states (TMSS) with squeezing parameters $r_1 \geq \dots \geq r_n \geq 0$.*

This means that in any orbit with respect to local Gaussian unitary operations there is a product state of n TMSS. When considering entanglement transformations with two pure Gaussian states ρ and ρ' of $n \times n$ modes, we can consider them without loss of generality to be of this normal form, characterized by vectors r and r' of ascendingly ordered squeezing parameters, respectively. We say that

$$r \geq r' \tag{9}$$

iff $r_k \geq r'_k$ for all $k = 1, \dots, n$, which allows the concise statement of our main result announced before.

Theorem 1 (Pure-state entanglement transformations) *Let ρ and ρ' be two $n \times n$ pure Gaussian states with CM γ and γ', characterized by ordered squeezing vectors r and r',*

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[a]Note that the $n \times m$ case is immediately included in the following considerations by adding $|n - m|$ vacuum modes to the side possessing fewer modes.

respectively. Then ρ can be transformed into ρ' by local Gaussian operations with classical communication iff $r \geq r'$, abbreviated as

$$\rho \longrightarrow \rho' \ \text{under} \ GLOCC, \ \text{iff} \ r \geq r'. \tag{10}$$

The subsequent Lemmas prepare the proof. Lemma 2 is a tool that connects a relation between two positive symmetric matrices to a relation involving the symplectic spectrum of the matrices.

Lemma 2 (Symplectic spectrum) *Consider real positive $2n \times 2n$ matrices $M_1, M_2 \geq 0$. Let $s(M_1)$ and $s(M_2)$ be the vectors consisting of the (ascendingly ordered) symplectic eigenvalues of M_1 and M_2, respectively. Then*

$$M_1 \geq M_2 \Longrightarrow s(M_1)_k \geq s(M_2)_k \quad \forall\, k = 1, \dots, n. \tag{11}$$

Proof: : As a consequence of $M_1 \geq M_2$ and that σ is a skew symmetric matrix we find $-\sigma M_1 \sigma \geq -\sigma M_2 \sigma$. As M_1 is a strictly positive real symmetric matrix, $M_1^{1/2}$ is well-defined and symmetric, and hence,

$$-M_1^{1/2} \sigma M_1 \sigma M_1^{1/2} \geq -M_1^{1/2} \sigma M_2 \sigma M_1^{1/2}, \tag{12}$$

and likewise $-M_2^{1/2} \sigma M_1 \sigma M_2^{1/2} \geq -M_2^{1/2} \sigma M_2 \sigma M_2^{1/2}$. From the corollary of Weyl's theorem known as monotonicity theorem [21] it follows that

$$\lambda_k \big(-M_j^{1/2} \sigma M_1 \sigma M_j^{1/2}\big) \geq \lambda_k \big(-M_j^{1/2} \sigma M_2 \sigma M_j^{1/2}\big) \tag{13}$$

for all $j = 1, 2$ and $k = 1, \dots, n$, where $\lambda(M)$ for any real symmetric matrix M is the vector of (ascendingly ordered) eigenvalues. But

$$\begin{aligned} \lambda_k \big(-M_1^{1/2} \sigma M_2 \sigma M_1^{1/2}\big) &= \lambda_k \big(-M_2^{1/2} \sigma M_1 \sigma M_2^{1/2}\big) \\ &\geq \lambda_k \big(-M_2^{1/2} \sigma M_2 \sigma M_2^{1/2}\big). \end{aligned} \tag{14}$$

Hence,

$$\lambda_k(-M_1 \sigma M_1 \sigma) \geq \lambda_k(-M_2 \sigma M_2 \sigma) \tag{15}$$

for all $k = 1, \dots, n$, which implies the validity of the right hand side of Eq. (11). ∎

Lemma 3 shows that we can restrict our considerations to the class of one-local pure transformations. These are then shown in Lemma 4 to have a simple form from which the desired result can be read off.

First, note that it does not restrict generality to impose the condition that in the course of the protocol the joint state is pure at all stages. This means that the CM associated with the completely positive map realizing the protocol can be taken to be of direct sum form. Moreover, as in the finite-dimensional case, one does not have to consider all local operations with classical operations, but only those with one-way classical communication:

Lemma 3 (One-local operations are sufficient) *Given two $n \times n$ pure states ρ and ρ' with CM γ and γ', respectively. Then ρ can be transformed into ρ' under GLOCC, iff ρ can be transformed into ρ' by one-local Gaussian operations (i.e., Gaussian local operations in system A with communication from system A to B only together with local Gaussian unitary operations in B).*

Proof: The analogous statement in the finite-dimensional setting has been proven in Ref. [22]: For general entanglement transformations of finite-dimensional pure states it does not restrict generality to make use of one-local operations only. It remains to be shown that the construction of Ref. [22] can be performed in the infinite-dimensional setting for Gaussian operations: Every GLOCC that is associated with a pure Gaussian state can be conceived as a sequence of elementary steps. Each such elementary step consists of local Gaussian unitary operations in one system, local Gaussian measurements, and the communication of the classical outcomes. Hence, it only has to be shown that each Gaussian measurement in system B can be equivalently implemented by means of a Gaussian measurement in system A, accompanied by appropriate local Gaussian unitary operations in both A and B. To see that this is the case, note firstly that the Schmidt decomposition can be applied in this infinite-dimensional case. Secondly, the unitary operation mapping any pure Gaussian state onto its Schmidt decomposition is a Gaussian unitary operation, as can be inferred from Lemma 1. According to the argument of Ref. [22], therefore, it follows that for any Gaussian state vector $|\omega\rangle_{B'}$, any bi-partite Gaussian state vector $|\psi\rangle_{AB}$, any unitary $U_{BB'}$ corresponding to a Gaussian unitary transformation, and any Gaussian state vector $|\phi\rangle_{B'}$ (potentially corresponding to an improper state [23]), there exist unitaries $V_{AA'}$ and V_B such that

$$\langle\phi|_{B'}(1 \otimes U_{BB'})|\psi\rangle_{AB}|\omega\rangle_{B'} = \langle\phi|_{A'}(V_{AA'} \otimes V_B)|\psi\rangle_{AB}|\omega\rangle_{A'}. \tag{16}$$

The unitaries $V_{AA'}$ and V_B in turn also correspond to Gaussian unitary operations. Therefore, any Gaussian measurement in system B leads to the same final pure state as a Gaussian measurement in system A, followed by appropriate local Gaussian unitary operations in both parts. ∎

Ref. [13] gives the general form of a Gaussian local operation with classical communication. It is considerably simplified for the special case that a local operation is implemented in one of the two parts of the joint system only. Then Eq. (6) becomes

Lemma 4 (Unilateral transformations) *Let γ be a CM of a Gaussian state of an $n \times n$ mode system consisting of systems A and B, partitioned as*

$$\gamma = \left(\begin{array}{cc} A & C \\ C^T & B \end{array} \right). \tag{17}$$

The CM γ' after application of a general Gaussian local operation in system A characterized by a CM

$$\Gamma_A = \left(\begin{array}{cc} \Gamma_{1A} & \Gamma_{12A} \\ \Gamma_{12A}^T & \Gamma_{2A} \end{array} \right) \tag{18}$$

is given by a matrix of the form as in Eq. (17) with

$$A' = \Gamma_{1A} - \tilde{\Gamma}_{12A}(\tilde{\Gamma}_{2A} + A)^{-1}\tilde{\Gamma}_{12A}^T, \tag{19a}$$

$$B' = B - C^T(\tilde{\Gamma}_{2A} + A)^{-1}C, \tag{19b}$$

$$C' = \tilde{\Gamma}_{12A}(\tilde{\Gamma}_{2A} + A)^{-1}C. \tag{19c}$$

As an example, we now discuss the unilateral transformation that transforms a TMSS with squeezing parameter r into another TMSS with squeezing parameter $r' < r$. On the level of

covariance matrices, this transformation can be achieved by applying quantum operations in system A only. The Gaussian operation that realizes this map is associated with a 4×4 CM Γ_A, which is given by

$$\Gamma_A = \begin{pmatrix} A_{r''} & C_{r''} \\ C_{r''} & A_{r''} \end{pmatrix}, \tag{20}$$

where as before $A_{r''} = \cosh(r'')1$ and $C_{r''} = \sinh(r'')\Lambda$. The squeezing parameter $r'' \in [0, \infty)$ is defined via

$$\cosh(r'') = \frac{\cosh(r)\cosh(r') - 1}{\cosh(r) - \cosh(r')}. \tag{21}$$

Physically, this operation can be implemented in two steps: first, one implements an appropriate Gaussian unitary operation on both the single mode of system A and an additional vacuum field mode. This can be done by applying suitable linear optical elements. Then, in a second step, one realizes a homodyne detection in the additional field mode. When considering the second moments only as we have done throughout the paper, no action is needed in system B. In fact, the classical outcome needs to be communicated only to apply the appropriate local displacement in phase space in system B. Equipped with Lemmas 2-4 we can now prove the Theorem.

Proof of the Theorem: Let us without loss of generality assume that ρ already is of the normal form of Lemma 1. Define for a squeezing vector r the $2n \times 2n$ matrices $c(r)$ and $s(r)$ as

$$c(r) := \bigoplus_k \cosh(r_k)1_2, \;\; s(r) := \bigoplus_k \sinh(r_k)\Lambda. \tag{22}$$

According to Lemma 4 the general form of a CM after a one-local Gaussian operation is given by γ' partitioned as in Eq. (40), with

$$A' = \Gamma_{1A} - \tilde{\Gamma}_{12A}[\tilde{\Gamma}_{2A} + c(r)]^{-1}\tilde{\Gamma}_{12A}^T, \tag{23a}$$

$$B' = S^T \left(c(r) - s(r)[\tilde{\Gamma}_{2A} + c(r)]^{-1}s(r) \right) S, \tag{23b}$$

$$C' = \tilde{\Gamma}_{12A}[\tilde{\Gamma}_{2A} + c(r)]^{-1}s(r)S, \tag{23c}$$

where S is the symplectic transformation corresponding to the Gaussian unitary operation in system B. Again, the final state can be taken to be of normal form. Imposing the condition $A' = B' = c(r')$ and $C' = s(r')$ and solving for Γ yields

$$\Gamma_{1A} = s(r') \left(\tilde{S}^T[\tilde{S}^T c(r)\tilde{S} - c(r')]\tilde{S} \right)^{-1} s(r') + c(r') \tag{24a}$$

$$\Gamma_{12A} = s(r')\tilde{S}^{-1} \left[\tilde{S}^T c(r)\tilde{S} - c(r') \right]^{-1} s(r), \tag{24b}$$

$$\Gamma_{2A} = s(r) \left[\tilde{S}^T c(r)\tilde{S} - c(r') \right]^{-1} s(r) - c(r), \tag{24c}$$

where $\tilde{S}$ is the symplectic transformation $\tilde{S} := \Lambda S\Lambda$. From the expressions for Γ_{1A} and Γ_{2A} one finds that $\Gamma_A \geq i\sigma$ can only hold if $\tilde{S}^T c(r)\tilde{S} - c(r') \geq 0$, meaning that

$$s[\tilde{S}^T c(r)\tilde{S}] \geq s[c(r')], \tag{25}$$

for which by Lemma 2 it is necessary that $r \geq r'$. On the other hand, if $r \geq r'$ we can choose $S = 1$ and one finds by direct calculation that Γ_A then describes a product of two-mode squeezed states in standard form with squeezing vector r''; the k-th entry r_k'' is given by

$$\cosh(r_k'') = \frac{\cosh(r_k)\cosh(r_k') - 1}{\cosh(r_k) - \cosh(r_k')}. \tag{26}$$

This argument shows that the conditions are also sufficient for the transformation of the states under GLOCC. ∎

4   Comparison with the majorization criterion

The simplicity of the above criterion is quite astonishing, as compared to the majorization structure in finite-dimensions [11]. As a corollary, it follows that not only pure-state distillation is not possible, but in fact any pure-state collective Gaussian quantum operation. In particular, so-called *catalysis of entanglement manipulation*, as it has been studied in the finite-dimensional case [24], can not occur: the metaphor of catalysis refers to the effect that in finite dimensions, it can happen that $\rho \not\longrightarrow \rho'$ under LOCC, but

$$\rho \otimes \omega \longrightarrow \rho' \otimes \omega \text{ under LOCC} \tag{27}$$

for some appropriate catalyst state ω. It should moreover be mentioned that – as Gaussian transformations can be made deterministic [15] – there is no space for distinct criteria for the stochastic interconversion between states which is again in contrast to the finite dimensional case [25], and this case is also covered by the above Theorem.

To explore to what extent the restrictions of the Ineq. (10) arise from the limitation to local *Gaussian* operations and which remain even if *general* local operations are allowed, we apply the state-transformation criterion of Nielsen [11] to pure Gaussian states for two typical examples. Eq. (10) has two main features. First, it implies that with Gaussian operations one cannot "concentrate" two-mode squeezing, i.e., increase the largest two-mode squeezing parameter that is available. This becomes particularly clear when considering two-mode squeezed states ρ_r with CM γ_r, where $r \in [0, \infty)$. They correspond to state vectors

$$|\psi_r\rangle = \cosh^{-1}(r/2) \sum_{k=0}^{\infty} \tanh^k(r/2) |k\rangle_A |k\rangle_B, \tag{28}$$

where $\{|k\rangle : k \in \mathrm{N}\}$ denotes the Fock basis. E.g., the transformation

$$\rho_r^{\otimes n} \longrightarrow \rho_s \otimes \rho_0^{\otimes(n-1)}, \ r < s \tag{29}$$

is not possible with GLOCC, no matter how close s is to r or how large n is. Second, one cannot "dilute" two-mode squeezing., i.e., the final r-vector cannot have more non-zero entries (or indeed entries strictly larger than any given threshold $s_0 \geq 0$) than the initial one. In particular, it is impossible to locally implement

$$\rho_s \otimes \rho_0 \longrightarrow \rho_r \otimes \rho_r, \ r > 0 \tag{30}$$

with GLOCC no matter how large s or how small r are.

We show now that transformations of the kind (29) *can* be realized if *arbitrary* local operations (LOCC) are allowed and r is sufficiently large. This shows (not surprisingly) that LOCC is more powerful than GLOCC, even when both initial and final state are required to be Gaussian. On the other hand, we show that (30) is *not possible* even with general (not necessarily Gaussian) local operations accompanied with classical communication.

For arbitrary local operations the transformation properties between bipartite pure states are governed by their Schmidt coefficients: ρ can be transformed into ρ' by LOCC, iff the Schmidt coefficients of ρ are majorized by those of ρ' [27]. The ordered list of Schmidt coefficients of ρ_s is given by the vector m with components

$$m_k = (1 - \eta)\eta^k, \tag{31}$$

where $\eta := \tanh^2(s/2)$ and $k = 0, 1, ...$ The ordered list of Schmidt coefficients of the initial state $\rho_r \otimes \rho_r$ is the vector

$$l := (1 - \lambda)^2(1, \lambda, \lambda, \lambda^2, \lambda^2, \lambda^2, \lambda^3, \lambda^3, \lambda^3, \lambda^3, ...), \tag{32}$$

with $\lambda := \tanh^2(r/2)$. The transformation $\rho_r \otimes \rho_r \longrightarrow \rho_s \otimes \rho_0$ under LOCC is possible if and only if

$$\sum_{k=0}^{N} l_k \leq \sum_{k=0}^{N} m_k \tag{33}$$

for all $N = 0, 1, ...$ Obviously, if we allow for a summation over more than the first N positive terms on the left hand side of Ineq. (33) while still having the inequality satisfied, the transformation is also possible. So, certainly $\rho_r \otimes \rho_r \longrightarrow \rho_s \otimes \rho_0$ under LOCC holds if

$$
\begin{aligned}
(1 - \lambda)^2 \sum_{k=0}^{N}(k + 1)\lambda^k &= (1 - \lambda)^2 \frac{d}{d\lambda}\left(\lambda \frac{1 - \lambda^{N+1}}{1 - \lambda}\right) \\
&\leq (1 - \eta)\frac{1 - \eta^{N+1}}{1 - \eta}. \tag{34}
\end{aligned}
$$

Considering

$$f(\lambda, \eta, x) := (1 - \eta)\frac{1 - \eta^{x+1}}{1 - \eta} - (1 - \lambda)^2 \frac{d}{d\lambda}\left(\lambda\frac{1 - \lambda^{x+1}}{1 - \lambda}\right) \tag{35}$$

as a function of a real $x \in (1, \infty)$, it follows immediately from an elementary discussion of the behavior of f that pairs of $\lambda, \eta \in (0, 1)$ with $\eta > \lambda$ can be found such that $f(\lambda, \eta, x) \geq 0$ for all $x \in (1, \infty)$ (take, e.g., $\lambda = 0.1$, $\eta = 0.11$). Hence, for such pairs of λ, η, Ineq. (33) is satisfied for all $N = 0, 1, ...$ Therefore, $r, s \in [0, \infty)$ with $r < s$ can be found such that

$$\rho_r \otimes \rho_r \longrightarrow \rho_s \otimes \rho_0 \quad \text{under LOCC, but} \quad \rho_r \otimes \rho_r \not\longrightarrow \rho_s \otimes \rho_0 \quad \text{under GLOCC.} \tag{36}$$

This argument shows that in principle, by allowing for all LOCC, one may 'pump' entanglement from two two-mode squeezed states into one of the two-mode systems, at the expense of reducing the entanglement of the other system.

In the case of Eq. (30), however, even with LOCC one can do no more than with GLOCC. To see this, we only have to look at the sum of squares of the first $K = (N+1)(N+2)/2$ Schmidt coefficients of $\rho_r \otimes \rho_r$ for some $N \in \mathbb{N}$. We find

$$L_K := \sum_{k=0}^{K-1} l_k = (1-\lambda)^2 \sum_{k=0}^{N} (k+1)\lambda^k = 1 - \left[1 + \frac{N+1}{\cosh^2(r/2)}\right] \tanh^{2(N+1)}(r/2). \qquad (37)$$

This is to be compared with the sum of the first K Schmidt coefficients of $\rho_s \otimes \rho_0$,

$$M_K := \sum_{k=0}^{K-1} m_k = 1 - \tanh^{2(K+1)}(s/2). \qquad (38)$$

Noting that K grows quadratically in N, we see that for N large enough $M_K > L_K$ — no matter how large $r > 0$. Therefore,

$$\rho_s \otimes \rho_0 \not\longrightarrow \rho_r \otimes \rho_r \ \text{ under LOCC} \qquad (39)$$

for all $r, s > 0$. This transformation is in other words not even possible under general local operations, and Eq. (10) is no further restriction. This statement is indeed the analogue of the statement for finite-dimensional systems, that the Schmidt number cannot be increased by LOCC.

In turn, in the asymptotic limit of infinitely many identically prepared initial states such a dilution procedure becomes possible again under LOCC for an appropriate choice of $r, s > 0$. Then, the possibility of such a transformation is governed only by the von-Neumann entropies of the reduced states held by both parties. For GLOCC, such a dilution of two-mode squeezing stays impossible, even in the asymptotic limit.

One should keep in mind, however, that the bounds provided by general LOCC for Gaussian states are extraordinarily optimistic in any practical context, as general quantum operations are required that act in infinite-dimensional Hilbert spaces. Such general operations are certainly beyond all realistic assumptions concerning the set of feasible operations that are available in actual experiments. This argument nevertheless points towards the possibility of realizing non-Gaussian operations that map known Gaussian states onto Gaussian states. That such maps can have the power to distill quantum entanglement was shown in [26] for pure states.

5   Discussion and Conclusion

We have presented a general criterion for the possibility of transforming one pure Gaussian state of $n \times n$ modes into another by means of Gaussian local operations with classical communication. We have seen that all those states are locally equivalent to a tensor product of two-mode squeezed states. The corresponding squeezing vector r completely describes the non-local properties of the state, giving rise to a simple partial order on the set of pure Gaussian states. The only transformations possible by GLOCC are towards "smaller" states in the sense of this partial order. In particular, one cannot increase the largest amount of two-mode squeezing, nor the number of two-mode squeezed states present. Moreover, the criterion implies that there are no collective effects in Gaussian entanglement transformations since the best one can do is always to go to the normal form and then try to *individually*

transform the two-mode squeezed states. Finally, we have put the criterion into the context of the majorization criterion for general local operations with classical communication. A very useful generalization would be concerned with a full criterion for mixed Gaussian quantum states. In fact, some of the structure of the above proof remains true in the mixed-state case, however, the normal form of Lemma 1 is not available and the squeezing vector is no longer the only characteristic of those states. Moreover, for mixed states one-local operations are no longer sufficient. Nevertheless, this paper can hopefully contribute to paving the way towards such a general tool.

Acknowledgments

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23. In particular, this holds for the 'eigenstates' of the position operator, which are elements of the dual space of Schwartz space.
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27. Care must be taken when extending state transformation results to infinite dimensional systems. E.g., it is no longer true, that all states with equal Schmidt coefficients can be converted into each other by local unitaries, as the counterexample $\psi = \sum_n c_n \left|n\right\rangle \left|n\right\rangle, \psi' = \sum_n c_n \left|n+1\right\rangle \left|n+1\right\rangle$ demonstrates.

    Nevertheless, the majorisation criterion can be used in this infinite dimensional setting when "ρ can be converted into ρ' is now defined to mean that *there is a sequence of local transformations T_k that for each k map ρ to ρ'_k such that ρ'_k approaches ρ' in trace norm*; i.e., while it may not be possible to transform the states exactly, the target state can be approximated to arbitrary accuracy and with probability 1 via LOCC. In the finite dimensional case this definition of convertibility is equivalent to the usual one.

    Then the equivalence of the majorisation condition and convertibility can be formally derived for all norm-approachable states, e.g., by investigating sequences of the finite-dimensional pure states ρ_k and ρ'_k obtained by projecting ρ, ρ' onto the subspaces corresponding to the k largest Schmidt coefficients. Considering the finite case for all $k \in \mathrm{N}$, one immediately arrives at the majorisation criterion in the infinite-dimensional case.
28. Note that transformations that are both symplectic and orthogonal reflect all those unitary Gaussian operations that can be implemented by means of passive optical elements (beam splitters and phase shifts).

Appendix A: Proof of Lemma 1

Proof: We write the CM γ of the pure state ρ as

$$\gamma = \begin{pmatrix} A & C \\ C^T & B \end{pmatrix}. \tag{40}$$

Purity of the state ρ implies that $-(\gamma\sigma)^2 = 1$. We first make use of this condition to show

that the symplectic spectrum of A and B are identical. This implies that local operations allow for achieving

$$A = B = \bigoplus_k a_k 1_{2n_k}, \tag{41}$$

where $a_k \geq 1$ are the symplectic eigenvalues of A and n_k is their multiplicity. In a second step we show that without further changing A and B, we can transform C by local Gaussian operations into $\oplus_k (a_k^2 - 1)^{1/2} 1_{2n_k}$. Renaming $a_k = \cosh(r_k)$ with an appropriate $r_k \geq 0$, $k = 1, ..., n$, then proves the claim.

(a) All further arguments derive from the equality

$$\begin{pmatrix} A\sigma A\sigma + C\sigma C^T\sigma & A\sigma C\sigma + C\sigma B\sigma \\ C^T\sigma A\sigma + B\sigma C^T\sigma & B\sigma B\sigma + C^T\sigma C\sigma \end{pmatrix} = -1, \tag{42}$$

which holds for all covariance matrices of pure states by virtue of $-(\gamma\sigma)^2 = 1$. From the two diagonal blocks we obtain

$$A\sigma A\sigma^T = 1 + C\sigma C^T\sigma, \; B\sigma B\sigma^T = 1 + C^T\sigma C\sigma. \tag{43}$$

But the spectrum of the matrices on the left hand side is directly related to the symplectic spectrum of A and B respectively: the eigenvalues of $A\sigma A\sigma^T$ are the squares of the symplectic eigenvalues of A. Since the matrices on the right hand side have the same spectrum, it follows that A and B have the same symplectic spectrum. Hence, $A = B = \oplus_k a_k 1_{2n_k}$, without loss of generality.

(b) For the second part, observe from the off-diagonal blocks in Eq. (42) that $A\sigma$ anti-commutes with $C\sigma$ and $C^T\sigma$. From this and the positivity of the a_k one shows directly that C must be block-diagonal, with blocks corresponding to the a_k eigenspaces of A. Now consider Eq. (42) again for each such block separately. We denote the corresponding $m \times m$ CM of a pure state as γ', partitioned in block form as in Eq. (40). Then $A' = B' = a1$ for some $a \geq 1$. For the off-diagonal block C' we find

$$C'\sigma C'^T\sigma = (a^2 - 1)1, \; \sigma C'^T\sigma = C'^T, \tag{44}$$

which imply that $C'C'^T = (a^2 - 1)1$, from which it follows that $C' = (a^2 - 1)^{1/2}O$, where O is both symplectic and orthogonal and can be removed by a local unitary operation without affecting A'. Such transformations that are both symplectic and orthogonal correspond to a passive transformation [28]. ∎