Quantum Information and Computation, Vol. 3, No. 2 (2003) 115–120

GENERATION AND DEGREE OF ENTANGLEMENT IN A RELATIVISTIC FORMULATION

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Received October 15, 2002
Revised January 23, 2003

The generation of entangled states and their degree of entanglement are studied in a relativistic formulation for the case of two interacting spin-1/2 charged particles. In the realm of quantum electrodynamics, we revisit the interaction that produces entanglement between the spin components of covariant Dirac spinors describing the two particles. In this way, we derive the relativistic version of the spin-spin interaction, widely used in the nonrelativistic regime. Following this consistent approach, the relativistic invariance of the generated entanglement is discussed.

Keywords: Entanglement, Special Relativity

Communicated by: R Jozsa & G Milburn

The generation of entangled states and the measurement of their degree of entanglement are at the heart of diverse fundamental tests of quantum mechanics. On the other hand, special relativity is a fundamental theory that has to be considered in the study of measurements realized by different moving observers. In special relativity, for example, the observed simultaneity of two space-like separated events can be broken when they are observed in a different reference frame. As a consequence, there is a natural interest in studying nonlocal quantum correlations in the framework of special relativity [1]. Recent experiments [2] have addressed the question about the compatibility of the apparently independent predictions of quantum mechanics and special relativity. In Ref. [3], Peres *et al.* have shown that the spin entropy of a single free spin-$\frac{1}{2}$ particle has no invariant meaning when the kinematical degrees of freedom are traced out, while in [4] a kinematical study of entangled states of two electron spinors are extensively studied. More recently, in Ref. [5], Gingrich and Adami have studied the transfer of entanglement between momentum and spin of two particles under a Lorentz transformation.

In this paper, a consistent approach is presented to the generation and measurement of entanglement in a quantum relativistic framework. As an illustrative example, the case of two interacting spin-1/2 massive particles, say electrons, is considered in the context of quantum electrodynamics (QED). We give a pedagogical derivation of the Breit interaction that produces entanglement [6, 7], maximal or not, between the spin components of the covariant Dirac spinors associated with the two particles. The main goal is to keep throughout the process a relativistic covariant formalism for the electrons and to deduce the quantum interactions that generate entanglement between their Dirac spinors. This interaction is the relativistic version of the spin-spin interaction studied in a nonrelativistic context, typically in quantum optics, solid state, and other fields. Revisiting the dynamical aspects of relativistic entanglement will permit us to discuss consistently the invariant properties of the degree of entanglement of the generated states [4] and to suggest experimental settings for testing entangling procedures in the relativistic domain such as pair creation in high energy physics.

Let us consider two non-relativistic spin-1/2 charged particles, e.g. electrons, 1 and 2 with charge e and spin states given in the basis of the z-axis eigenstates $|\uparrow\rangle = (1,0)^T$ and $|\downarrow\rangle = (0,1)^T$. For the system of the two-particle spin states we construct the spin tensor product states $|ij\rangle = |i\rangle \otimes |j\rangle$ with $i,j = \{\uparrow,\downarrow\}$. The magnetic dipole-dipole interaction between the spins of two particles, in nonrelativistic quantum mechanics, is described by the Hamiltonian

$$\begin{aligned} H &= -\boldsymbol{\mu}_1 \cdot \boldsymbol{\nabla} \times \left(\boldsymbol{\mu}_2 \times \boldsymbol{\nabla}\frac{1}{4\pi r}\right) \\ &= \frac{3(\mathbf{n}\cdot\boldsymbol{\mu}_1)(\mathbf{n}\cdot\boldsymbol{\mu}_2) - \boldsymbol{\mu}_1\cdot\boldsymbol{\mu}_2}{4\pi r^3} + \frac{2}{3}\boldsymbol{\mu}_1\cdot\boldsymbol{\mu}_2\,\delta(\mathbf{r})\ , \end{aligned} \tag{1}$$

where r is the distance between the electrons, $\mathbf{n}$ is a unit vector in the direction of r, $\boldsymbol{\mu}= (e\hbar/2m_e c)\boldsymbol{\sigma}$ is the dipole moment operator of each electron and $\boldsymbol{\sigma} \equiv \mathbf{x}\,\sigma^x + \mathbf{y}\,\sigma^y + \mathbf{z}\,\sigma^z$ is the spin operator. When Hamiltonian (1) is applied to the initial state $|\downarrow\uparrow\rangle$, for non-overlapping particles and for $\mathbf{n} = \mathbf{n}_z$ it generates entanglement by means of the term $-\boldsymbol{\mu}_1\cdot\boldsymbol{\mu}_2/4\pi r^3$. After an interaction time t an entangled state is produced, which up to a global phase is given by

$$|\Psi^-\rangle = \cos\theta\,|\downarrow\uparrow\rangle - i\sin\theta\,|\uparrow\downarrow\rangle\ , \tag{2}$$

where $\theta = (2/\hbar)\int J dt$ and $J = -e^2\hbar^2/(16\pi m_e^2 c^2 r^3)$.

The dipole interaction model in nonrelativistic quantum mechanics involves only the spin degrees of freedom of the two particles and its compatibility with special relativity, frequently introduced *ad hoc*, should be proven from fundamental principles. In what follows, we revisit the derivation of the spin interaction Hamiltonian, similar to Eq. (1), from first principles and in the realm of quantum electrodynamics, which describes the interaction in a relativistic manner. This allows the creation of entangled states, similar to Eq. (2), between the spin components of the Dirac spinors of the two particles, while keeping relativistic covariance in a natural way.

In the relativistic quantum domain we consider two identical spin-1/2 charged particles described by Dirac spinors. They may be considered as relativistic quantum mechanical objects or as one-state particles of quantum field theory that are allowed to interact with a quantized electromagnetic field. Their interaction will be studied within the covariant formalism of

QED by considering scattering processes. In particular, the spin interaction Hamiltonian can be derived by calculating the amplitude that corresponds to Feynman diagrams describing the scattering of the charged particles when they exchange one virtual photon [8]. Two diagrams contribute to this amplitude as a consequence of the indistinguishability of the particles resulting from the fermion statistics (see Fig. 1).

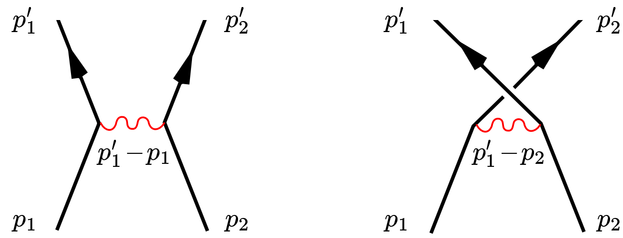

Fig. 1. The scattering process of two particles by the exchange of one virtual photon. This process incorporates the dipole-dipole interaction between two spin-1/2 charged particles.

Each particle participating in the scattering is described by a Dirac spinor, which for the plane wave case is given by [9]

$$\psi(x, \epsilon) = u(p, \epsilon) \mathrm{e}^{-\mathrm{i} p \cdot x} \ ,$$

where x is a point in space-time, p is the energy-momentum of the plane wave and ϵ is the polarization of the spin variable. The wave function $\psi(x, \epsilon)$ satisfies the Dirac equation

$$(\mathrm{i}\gamma^\mu \partial_\mu - m)\psi = 0 \ ,$$

where γ^μ are the Dirac matrices. Then, the vector $u(p, \epsilon)$ is a four dimensional spinor of the form

$$u(p, \epsilon) = \begin{pmatrix} \sqrt{p \cdot \sigma} \ \xi^\epsilon \\ \sqrt{p \cdot \bar{\sigma}} \ \xi^\epsilon \end{pmatrix} \ ,$$

where $\sigma^\mu \equiv (\mathbf{1}, \boldsymbol{\sigma})$, $\bar{\sigma}^\mu \equiv (\mathbf{1}, -\boldsymbol{\sigma})$ and ξ is any two component spinor normalized to $\xi^\dagger \xi = \mathbf{1}$. For example, for $\epsilon = \uparrow$, $\xi^\uparrow = (1, 0)^T$, which means that the particle has spin up in the z direction. The initial state of two well separated particles with known spin orientation can be expressed as

$$u(p_1, \epsilon_1) \otimes u(p_2, \epsilon_2) \ .$$

In the following, we calculate their scattering amplitude $\mathcal{M}$ without tracing their spin components, by employing the QED Feynman diagrams and rules. The leading order contributions to this amplitude in terms of a perturbation expansion of the QED coupling e are presented in Fig. 1. Considering that the coupling constant of the QED interactions is small, it is unlikely that two virtual photons are simultaneously exchanged between the electrons. Hence, the irreducible graphs given in Fig. 1 are describing efficiently the process even for higher energies [7]. The amplitude of this process is given by

$$i\mathcal{M} = (-ie)^2 \Big[\bar{u}(p_1', \epsilon_1')\gamma^\mu u(p_1, \epsilon_1) \frac{-ig_{\mu\nu}}{(p_1' - p_1)^2} \bar{u}(p_2', \epsilon_2')\gamma^\nu u(p_2, \epsilon_2)$$
$$- \bar{u}(p_1', \epsilon_1')\gamma^\nu u(p_2, \epsilon_2) \frac{-ig_{\mu\nu}}{(p_1' - p_2)^2} \bar{u}(p_2', \epsilon_2')\gamma^\mu u(p_1, \epsilon_1)\Big] . \quad (3)$$

For nonrelativistic velocities we can employ the Born approximation for the scattering amplitude, given by $\langle p'|i\mathcal{M}|p\rangle = -i\tilde{V}(\mathbf{q})\delta^{(3)}(E_{\mathbf{p}'} - E_{\mathbf{p}})$ for $\mathbf{q} = \mathbf{p}' - \mathbf{p}$. This does not violate the relativistic covariance of the description but rather isolates the dominant contributions for particles moving slowly in their center of mass reference frame [10]. By substituting for low momentum $u(p', \epsilon')\gamma^i u(p, \epsilon) \approx 0$ for $i = 1, 2, 3$ and $u(p', \epsilon')\gamma^0 u(p, \epsilon) \approx 1$ we obtain by Fourier transformation the usual Coulomb interaction $V(r) = e^2/(4\pi r)\delta^{\epsilon_1\epsilon_1'}\delta^{\epsilon_2\epsilon_2'}$. The Kronecker δ symbols indicate that the spin indices of the spinors remain unaffected by the Coulomb interaction. Here, only the first diagram has been employed as an antisymmetrization of the resulting wave function automatically compensates for the contributions due to the exchange diagram. In the next order of approximation with respect to small momentum, where still the Born approximation holds, we employ the exact formula

$$u(p_1', \epsilon_1')\boldsymbol{\gamma} u(p_1, \epsilon_1) = \xi^{\epsilon_1' \dagger} \left[-\frac{i(\mathbf{p}_1 + \mathbf{p}_1')}{2m} + \frac{\boldsymbol{\sigma}_1 \times (\mathbf{p}_1 - \mathbf{p}_1')}{2m} \right] \xi^{\epsilon_1} . \quad (4)$$

Considering in the amplitude of Eq. (3) only the cross terms with $\boldsymbol{\sigma}$, we obtain by Fourier transformation the dipole-dipole interaction matrix

$$-\frac{e\,\boldsymbol{\sigma}_1}{2m} \cdot \boldsymbol{\nabla} \times \left(\frac{e\,\boldsymbol{\sigma}_2}{2m} \times \boldsymbol{\nabla} \frac{1}{4\pi r} \right) . \quad (5)$$

This term is analogous to Eq. (1), giving rise to the spin interaction, but this time in the space of spinors.

If we have an initial two-particle state with non-overlapping wave functions and spins oriented oppositely along the z axis, the effective Born potential becomes

$$V = J\left(2\delta^{\epsilon_1\epsilon_2'}\delta^{\epsilon_2\epsilon_1'} - \delta^{\epsilon_1\epsilon_1'}\delta^{\epsilon_2\epsilon_2'} \right) , \quad (6)$$

equivalent to the term $-\boldsymbol{\mu}_1 \cdot \boldsymbol{\mu}_2/4\pi r^3$ of Eq. (1) but acting on four-component Dirac spinors. If the relativistic description of the particles is relaxed, the spin interaction of Eq. (1) is recovered.

We consider now the initial wave function of the two particles $u(p_1, \epsilon_1) \otimes u(p_2, \epsilon_2)$, which for simplicity we shall take to have zero three-momentum [11] and with spin components oriented oppositely along the z axis, i.e. $\epsilon_1 =\downarrow$ and $\epsilon_2 =\uparrow$. The evolved state after an interaction time t reads

$$\cos\theta\ u(\mathbf{p}_1 = 0, \epsilon_1 =\downarrow) \otimes u(\mathbf{p}_2 = 0, \epsilon_2 =\uparrow) - i\sin\theta\ u(\mathbf{p}_1 = 0, \epsilon_1 =\uparrow) \otimes u(\mathbf{p}_2 = 0, \epsilon_2 =\downarrow) , \quad (7)$$

up to an overall phase. Here, we have assumed that the particles are asymptotically at rest with respect to a common reference frame. The relativistic covariance of the entangled two-particle state of Eq. (7) is certainly not present in Eq. (2). Notice that the form of the coherent superposition of Eqs. (2) and (10) is preserved and the action integral $\int J dt$ is a relativistic invariant quantity due to the invariance of the action of the electromagnetic interactions [12].

The Dirac equation is relativistically covariant, i.e. it has an invariant form under Lorentz transformations from one inertial frame to the other. Let us consider a spin-1/2 particle with momentum p and spin ϵ described by the wave function $\psi(p,\epsilon)$. In a transformed reference frame its momentum is given by $p' = Lp$, where L is a Lorentz boost represented by a non-unitary matrix. The quantized Dirac states $|\psi(p,\epsilon)\rangle = \psi(p,\epsilon) a_{\mathbf{p}}^{\epsilon\,\dagger}|0\rangle$, with $a_{\mathbf{p}}^{\epsilon\,\dagger}$ being the creation operator of an electron and $|0\rangle$ the vacuum state, transform unitarily, i.e. $|\psi'(p',\epsilon')\rangle = U(L)|\psi(p,\epsilon)\rangle$.

The analytic form of each spinor with zero momentum in the center of mass frame of the two electrons is given by $u(\mathbf{p}=0,\epsilon=\downarrow\uparrow) = \sqrt{m}(\xi^{\downarrow\uparrow},\xi^{\downarrow\uparrow})^T$. According to the previous discussion, when the two particles are allowed to interact for a certain time such that $\theta = \pi/4$, the spinor "EPR state"

$$\Psi(\mathbf{p}_1 = 0, \mathbf{p}_2 = 0) = \frac{1}{\sqrt{2}}\Big[u(0,\epsilon_1=\downarrow)\otimes u(0,\epsilon_2=\uparrow) - iu(0,\epsilon_1=\uparrow)\otimes u(0,\epsilon_2=\downarrow)\Big]\,,$$

is generated. This state corresponds to a Lorentz frame where both particles are at rest with respect to each other. For this "EPR state" the kinematical degrees of freedom are incorporated in a relativistic formalism, in contrast to Eq. (2). Hence, the wave function of each of the particles can be transformed to a relativistic frame along the x direction where it becomes

$$u(\mathbf{p}=p_x\mathbf{x},\epsilon=\downarrow\uparrow) = \begin{pmatrix} \sqrt{E - p_x\sigma_x}\ \xi^{\downarrow\uparrow} \\ \sqrt{E + p_x\sigma_x}\ \xi^{\downarrow\uparrow} \end{pmatrix}\,.$$

Note that in general the kinematics and the spin degrees of freedom of each particle are not factorizable, so they have to be incorporated in the theory *ab initio* in the entanglement generation procedure. An arbitrary Lorentz transformation that consists of a boost and a rotation acting on the "EPR state" gives the state

$$\Psi(\mathbf{p}_1 = \mathbf{p}, \mathbf{p}_2 = \mathbf{p}) = \frac{1}{\sqrt{2}}\Big[u(\mathbf{p},\tilde{\epsilon}_1=\downarrow)\otimes u(\mathbf{p},\tilde{\epsilon}_2=\uparrow) - iu(\mathbf{p},\tilde{\epsilon}_1=\uparrow)\otimes u(\mathbf{p},\tilde{\epsilon}_1=\downarrow)\Big]\,, \qquad (8)$$

where $\tilde{\epsilon}$ indicates spin up or down in the new spin direction. Here, we are allowed to make this transformation as the spinors are written in a relativistic covariant formalism. The spin states of each particle observed in the moving reference frame are equivalent to a local unitary transformation of the spin states observed in the rest frame, thus preserving the degree of entanglement. Transforming the entangled state to a different reference frame results into a state that can be observed by moving detectors. Ideal detectors could distinguish the different orientations of the spin for different momenta of a particle as the momentum and the spin operators commute. Therefore, in the considered situation (see also Ref. [4]), even though the spin is not a relativistic invariant quantity, the degree of spin entanglement of two particles is. Note that in Ref. [3], the relativistic properties of the spin of a single particle with a momentum distribution was studied.

In principle, as we mentioned before, the standard high-energy experiments of pair creation or electron-electron scattering represent a natural realm, even though unusual, for testing these relativistic entangling properties. For the case of photons [4], the implementation of moving detectors in parametric down conversion experiments might be required.

In this paper, we reviewed the derivation of the spin-spin interaction employed in nonrelativistic quantum mechanics to describe the spin entanglement between two spin-1/2 charged particles from first principles, i.e. quantum electrodynamics. By employing the relativistic formalism of spinors the system can be described in a Lorentz covariant way throughout the whole process. Hence, we were able to produce entanglement between Dirac spinors and to ask consistently about its relativistic properties. We showed that, as a natural consequence, the measurement outcomes of any two moving observers should witness the same degree of entanglement, independent of their relative motion. In this relativistic context, it is clear that no superluminal communication or causality violation in the spin measurements is expected. All entangling Hamiltonians could be, in principle, derived from fundamental principles, showing the fundamental compatibility of quantum correlation measurements and special relativity.

Acknowledgements

We thank F. De Zela, C. Joachain, H. Walther, and N. Zagury for useful comments.

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11. The following consideration also holds for arbitrary initial well defined momentum states.
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