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TELEPORTATION AND DENSE CODING VIA A MULTIPARTICLE QUANTUM CHANNEL OF THE GHZ-CLASS

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A set of protocols for teleportation and dense coding schemes based on a multiparticle quantum channel, represented by the N-particle entangled states of the GHZ class, is introduced. Using a found representation for the GHZ states, it was shown that for dense coding schemes enhancement of the classical capacity of the channel due from entanglement is $N/N-1$. Within the context of our schemes it becomes clear that there is no one-to one correspondence between teleportation and dense coding schemes in comparison when the EPR channel is exploited. A set of schemes, for which two additional operations as entanglement and disentanglement are permitted, is considered.

Keywords: Entangled states, multipaticle quantum channel, teleportation, dense coding
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1. Introduction

In the quantum information processing the large number of protocols are based on a quantum channel involved entangled particles. In fact, an EPR pair is one of the main resources of two significant processes as quantum teleportation and dense coding, that are attractive not only from the theoretical point of view but for many applications. Dense coding introduced by Bennett et al [1] was realized in the optical experiment with polarized photons by Zeilinger et al [2] and for continuous variables by Peng et al [3]. The quantum teleportation protocol proposed by Bennett et al [4] has been implemented by several groups [5, 6, 7, 8].

A lot of teleportation and dense coding schemes and its applications have been considered by many authors both for discrete and continuous variables. Recently an approach for classification all schemes has been made by Werner [9]. Generally, this problem is very difficult and the main results have been obtained in the case of so-called *tight* schemes which are realized with minimal resources with respect to the Hilbert space dimensions and classical information. As result, it has been found a one-to-one correspondence between all *tight* schemes of teleportation and dense coding. Note, an EPR pair or a two-particle quantum channel is used

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in the *tight* schemes. The obtained result is significant for practice because it tells, that if one can teleport a qubit, then he could perform dense coding using the same experimental arrangement without any additional resources.

In this paper a quantum channel presented by a multiparticle entangled state of the GHZ class is considered for teleportation and dense coding. When the channel involves more then two particles, its features become quite complicated and all schemes are not *tight.* For particular case, one finds the GHZ channel based on the triplet of the GHZ form. If we wish to exploit the GHZ channel for teleportation, for example, then the task cannot be simply accomplished by a generalization of the usual protocol. Considering how to transmit an unknown qubit by the GHZ channel, Karlsson et al have shown, that the unknown state can be recovered by one of the two receivers, but not both [10]. With the use of the GHZ channel, the conditional teleportation of two entangled qubits has developed [14, 15]. What kind of a two-qubit state can be perfectly teleported via the GHZ channel? This problem has been considered by Marinatto et al [11]. It has been found that the general two-qubit state can't be transmitted perfectly but a pure entangled states can. This conclusion is in an agreement with our result presented in Ref. [12]. It should be noted also that recently Cereceda [16] considered the three-qubit dense coding scheme based on the GHZ channel and his results are in agreement with our protocol for the dense coding presented in ref.[17].

The main goal of this paper is to consider the multiparticle quantum channel of the GHZ class for informational tasks such as teleportation and dense coding. We introduce the protocol for the three-qubit dense coding and a scheme for distributing a mixed qubit state with two parties is discussed. The set of the questions we study in this paper is the following: what is the multiqubit dense coding schemes, whether we could have an enhancement of the channel capacity and whether the teleportation resources could be used directly for dense coding similarly *tight* schemes. Also we consider what kind of teleportation and dense coding schemes can be created using certain additional resources such as entanglement and disentanglement operations.

The paper is organized as follows. First, we discuss the main resources and consider *tight* schemes, then teleportation and dense coding protocols are introduced for the GHZ channel and a telecloning scheme is presented. It should be noted that we consider only some new additional possibilities of GHZ channel to produce an ideal copies of a mixed state. In the next section using the found representation of N particle entangled states we establish some main features of the multiparticle channel and calculate its capacity. Then new collection of teleportation and dense coding schemes is introduced, when such operations as entanglement and disentanglement are permitted.

2. Tight schemes

Following to Werner [9], we consider a set of objects to create some teleportation and dense coding schemes. The set includes an observable F, a collection of unitary operators T, an entangled state ω to be a quantum channel. Let the Hilbert spaces of the involved systems have the same dimension d, and ω is the N particle entangled state. Two parameters d and N play the key role. If $N = 2$ one can find schemes called *tight.*

Let the observable F be a complete set of the N- particle states, $\sum_x F_x = 1$, $F_x = |\Phi_x(N)\rangle\langle\Phi_x(N)|$, where x is one of the d^N elements of an output parameter space $X(d^N)$.

In general these pure states can be not maximally entangled. We assume, that T is the collection of the m - particle unitary operators $U_x(m)$, completely positive, which transform input state of the channel ω to output state $U_x(m)\omega U_x^\dagger(m)$. Let the N - particle quantum channel $\omega = |\Omega\rangle\langle\Omega|$ be shared N parties $A, B, C, ..$, spatially separated, where Ω can be one of the states of $\Phi_x(N)$. Then all resources are

$$R = \{\omega, x \in X(d^N), \Phi_x(N), U_x(m)\}. \tag{1}$$

Using (1) the teleportation and dense coding schemes could be obtained. We consider only the qubit case, for which $d = 2$. Note, the multiparticle quantum channel has new properties due from operators $U_x(m)$. When $m \geq 2$, these operators may be non-local in contrast the *tight* schemes.

If $N = 2$, one finds an EPR channel of the form $\Omega = (|00\rangle + |11\rangle)/\sqrt{2}$, that is shared two parties, Alice and Bob. Here the observable F is described by the Bell states $\Phi_x(2) = \Phi^+, \Psi^+, \Phi^-, \Psi^-$, the set of unitary operators consists of the Pauli and the identity operators $U_x(1) = \mathbb{1}, \sigma_z, \sigma_x, -i\sigma_y$ and the space X has four elements $x = 0, 1, 2, 3$ by which the 2 bits of information can be encoded. The following map is possible

$$x \leftrightarrow \Phi_x(2) \leftrightarrow U_x(1) \tag{2}$$

Reading (2) as $\Phi_x(2) \mapsto x \mapsto U_x(1)$ one finds teleportation that allows Alice sending to Bob an unknown qubit $\zeta = \alpha|0\rangle + \beta|1\rangle$, where $|\alpha|^2 + |\beta|^2 = 1$. In accordance with the teleportation protocols, Alice performs the Bell state measurement on her half of ERP pair and the unknown qubit. Outcomes of the measurement x can be encoded with the use of the unitary operators $U_x(1)$ by which Bob acts on his half of EPR pair to recover the unknown state. One ERP pair and 2 bits of classical information are needed for teleporting a single qubit.

Reading (2) as $x \mapsto U_x(1) \mapsto \Phi_x(2) \mapsto x$, one can find the dense coding scheme, that permits Alice sending of a two-bit message to Bob, manipulating one bit only. Because of 2 bits of information $\tilde{x} = 00, 01, 10, 11$ can be encoded by the four operators $U_x(1)$, Alice can generate the Bell basis manipulating her particle of EPR pair

$$\Phi_x(2) = (\mathbb{1} \otimes U_x(1))\Omega. \tag{3}$$

Then 2 bits of information are storied in four orthogonal states, that can be distinguished, if Bob performs the Bell state measurement. The properties of this channel are described by the Holevo bound, that tells us that the classical capacity of this quantum channel increases twice because of entanglement.

The considered schemes of perfect teleportation and dense coding are called *tight* [9] in the sense of the required resources. These resources are: the EPR channel, the 2 bits of information, the Bell state measurement and a collection of the one-particle unitary operators. Werner has proved a theorem, that for all tight schemes there is a one-to-one correspondence between teleportation and dense coding.

3. The GHZ channel

If three-particle entanglement is used instead of EPR pair one finds a channel which features are more complicated. This channel shared multi users, Alice, Bob and Claire, allows not only transmitting of quantum state and classical information by teleportation and dense coding, but distributing quantum states between several parties by copying or telecloning.

There is a complete set of the three particle entangled states of the form

$$(|000\rangle \pm |111\rangle)/\sqrt{2}, \quad (|001\rangle \pm |110\rangle)/\sqrt{2}$$
$$(|010\rangle \pm |101\rangle)/\sqrt{2}, \quad (|011\rangle \pm |100\rangle)/\sqrt{2}. \tag{4}$$

Without loss of generality a triplet of the GHZ form can be chosen as the quantum channel, whose three particles A, B and C are shared Alice, Bob and Claire. Then $\Omega = |GHZ\rangle$, where

$$|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)_{ABC}. \tag{5}$$

All schemes, based on the GHZ channel, are not tight and the Werner's theorem cannot guarantee the one-to-one correspondence between teleportation and dense coding schemes.

3.1. *Teleportation*

One of the main features of the GHZ channel is a perfect transmitting of the two particle entangled states of the EPR form $\zeta = \alpha|01\rangle + \beta|10\rangle$, where $|\alpha|^2 + |\beta|^2 = 1$. It has been shown that the general state of a pair of qubits cannot be transmitted through the GHZ channel[11].

If Alice wishes to send the entangled state ζ of the qubit 1 and 2, she needs to perform the measurement on particles 1, 2 and her particle A of the GHZ channel in the basis $\Phi_x(3)$ of the form [12]

$$\Phi_x(3) = \{\pi_1^\pm \otimes \Phi_{2A}^\pm, \pi_1^\pm \otimes \Psi_{2A}^\pm\}, \tag{6}$$

where $\pi^\pm = (|0\rangle \pm |1\rangle)/\sqrt{2}$. Then the total state is the product $|\zeta\rangle_{12} \otimes |GHZ\rangle_{ABC}$. The task is accomplished because of all outcomes of the measurement have equal probabilities are non depended from the unknown state and there is a set of the two-particle unitary operators $U_x(2)$, that allow receivers recovering the state to be teleported. It is important, that all operators can be chosen in the factorized form

$$U_x(2) = B_x \otimes C_x, \tag{7}$$

where B_x and C_x are the Pauli operators, that act on the Bob and Claire particle. In more details it reads

x	$\Phi_x(3)$	$\|BC\rangle_x$	B_x	C_x
0	$\pi^+ \otimes \Phi^+$	$\beta\|00\rangle + \alpha\|11\rangle$	σ_x	1
1	$\pi^+ \otimes \Phi^-$	$\beta\|00\rangle - \alpha\|11\rangle$	$i\sigma_y$	1
2	$\pi^- \otimes \Phi^+$	$-\beta\|00\rangle + \alpha\|11\rangle$	$-i\sigma_y$	1
3	$\pi^- \otimes \Phi^-$	$-\beta\|00\rangle - \alpha\|11\rangle$	$-\sigma_x$	1
4	$\pi^+ \otimes \Psi^+$	$\beta\|11\rangle + \alpha\|00\rangle$	1	σ_x
5	$\pi^+ \otimes \Psi^-$	$\beta\|11\rangle - \alpha\|00\rangle$	1	$-i\sigma_y$
6	$\pi^- \otimes \Psi^+$	$-\beta\|11\rangle + \alpha\|00\rangle$	1	$i\sigma_y$
7	$\pi^- \otimes \Psi^-$	$-\beta\|11\rangle - \alpha\|00\rangle$	1	$-\sigma_x$

(8)

It follows from (8), that Alice cannot rotate the qubit of Bob and vice versa, but transformations are correlated because of they have the same indexes. Thus operators U_x can be considered as local ones. In the same time the task is accomplished, if operators $U_x(2)$ are not factorized. Let $\Phi_x(3) = |\pi^+\rangle_1|\Phi^+\rangle_{2A}$, then one finds two operators $\sigma_x \otimes \mathbb{1}$ and $U_{BC} = (\sigma_x \otimes \mathbb{1})C_{BC}C_{CB}C_{BC}$ be sufficient, where C_{ct} is CNOT gate, c is a control bit, t is a target bit, $c, t = B, C$. Indeed, the introduced operation U_{BC} is non-local, that is the feature of the GHZ channel. The GHZ channel shared three parties A,B and C, spatially separated, allows distributing information with B and C. Two ideal copies of a mixed state could be produced by a teleportation protocol so that we shall call it telecloning. We consider a scheme, whose main resources are the Bell state measurement and set of the Pauli operators as for *tight* schemes.

Let using the above resources Alice wishes to send to Bob and Claire an unknown qubit in a mixed state

$$\rho_1 = \lambda_0|0\rangle\langle 0| + \lambda_1|1\rangle\langle 1|, \tag{9}$$

where $\lambda_0 + \lambda_1 = 1$. Then combined density matrix is $\rho_1 \otimes |GHZ\rangle_{ABC}\langle GHZ|$. The Bell state measurement on the qubit 1 and A projects the particle B and C into the state, that has the form $\rho_{BC} = \lambda_0|bb\rangle\langle bb| \pm \lambda_1|\bar{b}\bar{b}\rangle\langle\bar{b}\bar{b}|$, where $b = 0, 1$, $\bar{b} = 1 - b$. Two bits of information allow Bob and Claire to obtain the density matrix

$$\rho'_{BC} = \lambda_0|00\rangle\langle 00| + \lambda_1|11\rangle\langle 11|. \tag{10}$$

It is a separable and classically correlated state of two particles. In the same time both receivers have in their hands the unknown state, since the reduced matrices read $\rho_B = \rho_C = \lambda_0|0\rangle\langle 0| + \lambda_1|1\rangle\langle 1|$. As result, two perfect copies of an unknown mixed state are prepared by teleportation. In fact, these copies are not independent, that follows from the no-cloning theorem. However each receiver can manipulate his state independently, if and only if he performs local unitary operations. It is not true, when one of them decides to make a measurement on his own particle. Note that we have no had any aim to consider the problem of cloning of unknown quantum state in its general form, but considered only some new additional possibilities of GHZ channel to produce an ideal copies of a mixed state. The considered telecloning protocol could be useful in practical realization, for example, as a pumping mechanism (see [20, 21]).

3.2. *Dense coding*

Is it possible to use the teleportation resources given by (6) and (7) for dense coding similar the case of the EPR channel? The answer is not, however, a scheme of dense coding can be achieved.

Let a sender wishes to transmit a three bit message. The 3 bits of information 000, ..., 111 can be encoded by a set of the eight states D_x each of which is obtained from the GHZ state using a collection of the unitary operators $U_x(2)$ in accordance with (3), for example. Let the two particle operators be chosen in the factorized form (7), then equation (3) reads

$$|D_x\rangle_{ABC} = \mathbb{1} \otimes B_x \otimes C_x|GHZ\rangle_{ABC}. \tag{11}$$

An appropriate collection of the Pauli operators permits a sender to generate the complete set of states D_x, given by (4). All these states are well distinguishable by measurement, which outcomes encode the three bit message. Then one finds a dense coding scheme, that is described by the map of the form

$$\begin{array}{cccc}
\tilde{x} & B_x & C_x & D_x = B_x C_x|GHZ\rangle_{ABC} \\
000 & \mathbb{1} & \mathbb{1} & |000\rangle + |111\rangle \\
001 & \mathbb{1} & \sigma_x & |001\rangle + |110\rangle \\
010 & \sigma_z & \mathbb{1} & |000\rangle - |111\rangle \\
011 & \sigma_z & \sigma_x & |001\rangle - |110\rangle \\
100 & \sigma_x & \mathbb{1} & |010\rangle + |101\rangle \\
101 & \sigma_x & \sigma_x & |011\rangle + |100\rangle \\
110 & -i\sigma_y & \mathbb{1} & |010\rangle - |101\rangle \\
111 & -i\sigma_y & \sigma_x & |011\rangle - |100\rangle,
\end{array} \tag{12}$$

where we have omitted the normalization factor $1/\sqrt{2}$ in D_x. Equation (12) presents the protocol of the dense coding schemes based on the GHZ channel.

To find the capacity of this GHZ channel it needs to calculate the Holevo function

$$C(\{p_i\}, \rho) = S(\rho) - \sum_i p_i S(\rho_i), \tag{13}$$

where $\rho = \sum_i p_i \rho_i$, ρ_i are the density matrices of the states sent to receiver according to probabilities p_i and $S(\rho)$ is the von Neumann entropy. For the considered case $\rho_i = |D_i\rangle\langle D_i|$ and the channel is represented by the maximally entangled state, then assuming $p_i = 1/8$, one finds $C = S(\sum_i |D_i\rangle\langle D_i|/8) = 3$, hence per transmitted bit $C/2 = 3/2$, that is the classical capacity of the quantum channel. It means, the channel capacity due from entanglement increases in 3/2 times. Evidently, this result is clear without any calculations. Because of the presented protocol allows sending the 3 bits of information manipulating two bits only, as result, profit is 3/2, which is enhancement of the capacity. Also it is clear, that it results from the entanglement, which degree has to be maximum.

Inspection of (6) and (7) shows, that the teleportation resources are inapplicable for dense coding. The reason is that collection of states D_x, obtained in accordance with equation (11), where operators C_x, B_x are given by (7), is not the complete set. Also, being suitable for dense coding the complete set given by (4) cannot be used for teleportation, because of the outcomes of measurement depend on the unknown state. Therefore there is not a one-to one correspondence between these schemes in contrast the *tight* schemes. However a connection can be established. Indeed, two sets namely D_x given by (7) and $\Phi_x(3)$ denoted by (6) can be converted from one to another using an unitary operation, say of the form $\Phi_x = H_B C_{BC} D_x$, where H_B is the Hadamard transformation of particle B. It follows from (11), that eight distinguishable states can be generated by manipulating only two bits of the GHZ channel as follows

$$\Phi_x(3) = H_B C_{BC}(\mathbb{1} \otimes B_x \otimes C_x)|GHZ\rangle, \tag{14}$$

where B_x, C_x are given by (12).

Equation (14) tells that the measurement from the teleportation scheme may be used for dense coding. For that it needs to replace the operations $B_x \otimes C_x \to H_B C_{BC}(B_x \otimes C_x)$ before sending the message. Then 3 bits of information are stored in the complete set of states to be well distinguishable by the projective measurement of Φ_x. Note, the unitary operations become non-local, that is the one of the particular qualities of the three-particle channel.

Indeed, for the dense coding schemes the GHZ channel could be created by operations $U_x(2)$. Let only two qubits A and B of the three ones A, B and C be entangled, in other words the EPR channel and the ancilla qubit C are introduced, then the GHZ state is prepared by the way $C_{BC}(|\Phi^+\rangle_{AB} \otimes |0\rangle_C) = |GHZ\rangle_{ABC}$. This transformation can be inserted into each unitary operator $U_x(2)$, that becomes more complicated because of $U_x(2) \to U_x(2)C_{BC}$. In the same time it looks as the EPR pair is used instead of the GHZ channel.

4. The N-particle quantum channel

Some main features of the teleportation and dense coding schemes can be summarized when considering a multi particle channel for which the following mapping plays the key role

$$x \leftrightarrow U_x(m) \leftrightarrow \Phi_x(N). \tag{15}$$

By contrast the *tight* schemes, it seems to be a hard problem to prove it generally, therefore we will restrict ourselves by several cases.

4.1. *Representation for multiparticle states of the GHZ class*

Considering the resources given by (1) one finds the factor m in the operator $U_x(m)$ to be important, as it might be noticed from the case of GHZ channel. For teleportation schemes m is a number of particles on which an unknown state is transmitted, in other words, m shows how many particles can be teleported by the channel. For dense coding m indicates a number of particles for manipulating to send the N bit message and ratio N/m becomes the classical capacity of the quantum channel due from entanglement.

A one-to-one correspondence $x \leftrightarrow \Phi_x(N)$, where x is one of the 2^N outcomes of the von Neumann measurement, described by a complete but not over complete set of projectors, is clear. By contrast, the map $U_x(m) \leftrightarrow \Phi_x(N)$ is not so trivial. Similarly (3) one could write

$$\Phi_x(N) = (\mathbb{1}^{\otimes(N-m)} \otimes U_x(m))\Omega, \tag{16}$$

where $\mathbb{1}^{\otimes(N-m)}$ is product of $N-m$ identity operators $\mathbb{1} \otimes \mathbb{1}$.... According to the following rough dimension count, factor m can be established from (16). In fact, being the N qubit state, vector $\Phi_x(N)$ has the 2^N components. Any the m qubit operator $U_x(m)$ has the 2^{2m} matrix elements. Then for correspondence between $\Phi_x(N)$ and $U_x(m)$, it needs

$$m \geq \frac{N}{2}. \tag{17}$$

Indeed, these reasons are true not only in the qubit case, but for arbitrary dimension of the Hilbert space d.

A simple observation allows to obtain factor m with more accuracy. Let the channel be represented by a maximally entangled state Ω of the GHZ class

$$|\Omega\rangle = (|0\rangle^{\otimes N} + |1\rangle^{\otimes N})/\sqrt{2}, \tag{18}$$

where $|b\rangle^{\otimes N}$ is tensor product $|b\rangle \otimes \dots |b\rangle$, that is a state of the N independent qubits in the Hilbert space $\mathcal{H}_1 \otimes \dots \mathcal{H}_N$, $b = 0, 1$.

Proposition 1: *The set of the N particle vectors*

$$\Phi_{b_1 b_2 \dots b_N}(N) = (|0\rangle \otimes |b_2 \dots b_N\rangle + (-1)^{b_1}|1\rangle \otimes |\bar{b}_2 \dots \bar{b}_N\rangle)/\sqrt{2} \tag{19}$$

from the Hilbert space $\mathcal{H}_1 \otimes \dots \mathcal{H}_N$, *where* $b_1 = 0, 1$ $|b_2 \dots b_N\rangle = |b_2\rangle \otimes \dots |b_N\rangle$, *and* $|b_k\rangle = 0, 1$, $\bar{b}_k = 1 - b_k$ *is the orthonormal basis in* $\mathcal{H}_k$ *for each* $k = 2, \dots N$, *is the complete set of maximally entangled states.*

If $N = 2$, one finds the Bell states $\Phi_{b_1 b_2}(2) = (|0\ b_2\rangle + (-1)^{b_1}|1\ \bar{b}_2\rangle)$, that are generated by two classical bits $b_1, b_2 = 0, 1$. When $N = 3$ three bits $b_k = 0, 1$, $k = 1, 2, 3$ generate the set $D_x(3) = \Phi_{b_1 b_2 b_3}(3)$, given by (4). Also $\Omega = \Phi_{00\dots 0}(N)$ belongs to the collection (19).

Proof: Each of the states, that has the form (19), is maximally entangled in the sense of the reduced von Neumann entropy $E = S(\rho(1))$, where $\rho(1)$ is the one-particle density matrix. It follows from (19), that for any particle $\rho(1) = \mathbb{1}/2$, then $E = 1$, and entanglement is maximum. Also one finds the considered set of states to be complete, because of condition

$$\sum_{b_1 \dots b_N = 0,1} |\Phi_{b_1 b_2 \dots b_N}(N)\rangle\langle\Phi_{b_1 b_2 \dots b_N}(N)| = 1, \tag{20}$$

that directly results from the completeness of the collections $|b_k\rangle$.
Note, all possible entangled states can't be written in the form (19). It represents the GHZ like class only. For example, W-states, introduced by Cirac et al [18] and, so called Zero Sum Amplitude (ZSA) states, proposed by Pati [19], have another form and cannot be converted from the GHZ-class by local operations.

The next observation plays the key role. Equation (19) tells, that to generate all states of the set, it needs manipulating $N-1$ qubits of any fixed state from this collection. In other words, there is a set of operators $U_x(m)$, including identity operator, for which

$$m = N - 1. \tag{21}$$

It is in agreement with (17). It results in equation (19) takes the form of (16)

$$\Phi_{b_1 b_2 \dots b_N}(N) = (\mathbb{1} \otimes U_{b_1 b_2 \dots b_N}(N-1))\Omega, \tag{22}$$

where the string of bits $b_1 b_2 \dots b_N$ is binary notation of x, $x = 0, \dots 2^N - 1$.

Generally the question of existence and uniqueness of operators $U_{b_1 b_2 \dots b_N}(N-1)$ seems to be rather hard problem and we shall discuss simple examples. Let all operators be factorized and have the form of product of the one particle operators

$$U_{b_1 b_2 \dots b_N}(N-1) = U_{b_1 b_2}(1) \otimes U_{b_3}(1) \dots U_{b_N}(1). \tag{23}$$

Assume that each of the transformations $U_{b_1 b_2}(1), U_{b_3}(1) \dots$ can be represented by the Pauli operators. If $N = 2$ one finds $U_{b_1 b_2}(1) = \sigma_x^{b_2} \sigma_z^{b_1}$, and in accordance with (19) and (22)

$$(\mathbb{1} \otimes \sigma_x^{b_2} \sigma_z^{b_1})(|00\rangle + |11\rangle) = |0b_2\rangle + (-1)^{b_1}|1\bar{b}_2\rangle, \tag{24}$$

where $b_1, b_2 = 0, 1$. When $N > 2$, the choice $U_{b_k}(1) = \sigma_x^{b_k}$ for $k = 3, \dots N$ is suitable

$$(\mathbb{1} \otimes \sigma_x^{b_2} \sigma_z^{b_1} \otimes \sigma_x^{b_3} \otimes \dots \sigma_x^{b_N})\Omega = (|0b_2 \dots b_N\rangle + (-1)^{b_1}|1\rangle \otimes |\bar{b}_2 \dots \bar{b}_N\rangle)/\sqrt{2}. \tag{25}$$

The obtained equations (22) and (25) tell that the complete set of the N qubit entangled states of the GHZ class can be associated with a set of the $N-1$ particle operators, that generate all these states from one of them. In other words the mapping given by (15) can be justified. Indeed, the choice of operators may be not unique. For example, if $N = 3$, there is a case for which it is possible to manipulate one qubit instead of two qubits

$$(\mathbb{1} \otimes \sigma_z \otimes \sigma_x)(|001\rangle - |110\rangle) = (\mathbb{1} \otimes \mathbb{1} \otimes i\sigma_y)(|001\rangle - |110\rangle). \tag{26}$$

The representation given by (22) is not true for any states to be separable, it needs entanglement not less then two particles. A state of N's independent qubits can be written in the form $\Phi_{b_1 b_2 \dots b_N}(N) = |b_1 \dots b_N\rangle$. When bits take their value 0 and 1, the obtained set is complete, but it is important, that it can be generated from one of them by manipulating all qubits. Then instead of (22), one finds $\Phi_x(N) = U_x(N)\Omega$. If two qubits are maximally entangled and others are independent, then such state has the form $\Phi_{b_1 b_2}(2) \otimes |b_3 \dots b_N\rangle$, where $\Phi_{b_1 b_2}(2)$ is one of the Bell states. Entanglement allows to obtained complete set manipulating $N-1$ qubits of an initial state, say, $\Omega' = \Phi_{00} \otimes |0 \dots 0\rangle$. It is important, that Ω' does not belong to the GHZ class by contrast Ω, given by (18). From the physical point of view it is clear, that both states Ω' and Ω can't be transformed from one to another by local operations. For example, if $N = 3$ one finds transformation

$$(\mathbb{1} \otimes C_{23})\Phi_{00}(2) \otimes |0\rangle = |GHZ\rangle, \tag{27}$$

where $\Phi_{00}(2) = (|00\rangle + |11\rangle)/\sqrt{2}$. Here the CNOT operation C_{23} involves two qubits simultaneously, that is an interaction between two systems, that results in entanglement. When the GHZ state is prepared, as initial state Ω, the complete set can be obtained in accordance with (22), but the operators take the non-local form $U_{b_1b_2b_3} = (\mathbb{1} \otimes U_{b_1b_2}(1) \otimes U_{b_3}(1))(\mathbb{1} \otimes C_{23})$.

This example indicates the fact, that a complete set of the N qubit entangled states can be generated performing the non local operations on $N-1$ particles.

4.2. *Capacity of the channel*

Using (25), one finds a dense coding scheme, that allows sending a N-bit message by manipulating $N-1$ bits. To discuss capacity of the channel due from entanglement it needs to replace $\Omega \to \alpha|0\rangle^{\otimes N} + \beta|1\rangle^{\otimes N}$, where $|\alpha|^2 + |\beta|^2 = 1$. Now the channel is not assumed to be maximally entangled and its measure of entanglement, given by the reduced von Neumann entropy, has the form

$$E = -|\alpha|^2 \log|\alpha|^2 - |\beta|^2 \log|\beta|^2. \tag{28}$$

The Holevo function reads $C(\{p_x\}, \rho) = S(\rho/2^N)$, where all probabilities are equal and $p_x = 1/2^N$. For the considered channel

$$\rho = \sum_x (\mathbb{1}\otimes U_x(N-1))|\Omega\rangle\langle\Omega|(\mathbb{1}\otimes U_x(N-1))^\dagger = \sum_{b_1\ldots b_N=0,1} |\Phi'_{b_1b_2\ldots b_N}(N)\rangle\langle\Phi'_{b_1b_2\ldots b_N}(N)|. \tag{29}$$

Let operators $U_x(N-1)$ be factorized and have the form (25), then

$$\Phi'_{b_1b_2\ldots b_N}(N) = \alpha|0\rangle \otimes |b_2\ldots b_N\rangle + (-1)^{b_1}\beta|1\rangle \otimes |\bar{b}_2\ldots \bar{b}_N\rangle. \tag{30}$$

All these states are generated from Ω by the local unitary transformations, then their degree of entanglement is E, given by (51).

If $\alpha = \beta = 1/\sqrt{2}$, then $E = 1$ and the channel is maximally entangled. In the same time it implies the important fact, that the set of states becomes complete in accordance with (20) and all these states can be well distinguishable by measuring. For a maximally entangled channel the equation (29) is the condition of completeness and density matrix ρ takes the form $\rho = \rho(1)^{\otimes N}$, where the single particle density matrix is $\rho(1) = \mathbb{1}/2$.

When the channel can be not maximally entangled, one finds

$$\rho = \rho'(1) \otimes \rho(1)^{\otimes(N-1)}, \tag{31}$$

where $\rho'(1) = |\alpha|^2|0\rangle\langle 0| + |\beta|^2|1\rangle\langle 1|$ is the one-particle density operator. Before calculating the classical capacity of the channel, given by the Holevo function, note that it can be normalized per transmitted bit. For the considered protocol there are $N-1$ bits, that Alice transmits to Bob. Then using (31), capacity of the channel has the form

$$c = \frac{C(\{p_x\}, \rho)}{N-1} = 1 + \frac{E}{N-1}. \tag{32}$$

It takes maximum $c_{max} = N/N-1$, when $E = 1$. It means that entanglement results in increasing of the classical capacity of the N particle channel by $N/N-1$ times. Indeed, this result is clear without calculating. If a channel permits sending of N-bits of classical information manipulating $N-1$ qubits, then profit is $N/N-1$, that is enhancement of the channel capacity per transmitted bits.

4.3. *Sufficient tight and other schemes*

The main resources, given by (1), are sufficient also for teleportation of the entangled states of the form $\zeta = \alpha|0\rangle^{\otimes(N-1)} + \beta|1\rangle^{\otimes(N-1)}$. The task can be accomplished by the N particle channel Ω and the $N/N-1$ bits of classical information per transmitted particle because of the N-particle measurement. The measurement involves all particles to be teleported and one particle from Ω. It can be described by observable of the form (6), where $\pi^{\pm} \to (\pi^{\pm})^{\otimes(N-2)}$ [12, 13].

The presented teleportation and the dense coding schemes are based on the mentioned resources to be sufficient and minimal for these tasks. This set of schemes we shall name *sufficient tight* schemes by contrast the other ones, that can be obtained if some additional resources are permitted.

Let introduce the k-bit operators $En(k)$ and $Den(k)$ to be transformations of entanglement and disentanglement. Their existence is the open question generally, In our particular case we consider unitary operations, that can entangle and disentangle only a specific set of pure states. Let operators $En(k)$ and $Den(k)$ transform k independent qubits into a k-qubit entangled state and vice versa. For our purpose $C-NOT$ and Hadamard gates are useful. As for implementation of the En(2) operation one could bear in mind Optical Parametric Oscillator (OPO), well known in quantum optics. OPO generates EPR pair by "entangling" two input vacuum modes in the non-linear transparent crystal to be loss-free-medium in good approximation, therefore theoretically this transformation is unitary and reversible. Some modifications of schemes arise when these operations are permitted. It is well known that operator $En(2)$, say of the form $En(2) = C_{12}$, plays the key role in the one-bit teleportation, when an unknown qubit is entangled with ancilla [22]. It results in one bit of classical information is needed for sending the qubit. Indeed, the one-bit protocol can be directly generalized for teleportation of two entangled qubits, when two bits of information are required.

For dense coding schemes all modifications reduce to preparing of the channel state Ω and revising of observable just as the way the considered GHZ channel. Suppose, there is a collection of N qubits, in which the k particles are independent and the remainder $N-k$ qubits are entangled. Let only one qubit from entanglement be in the receiver hand. For preparing Ω it needs entanglement of all particles that can be achieved with the use of operator $En(k+1)$. It looks as all operators $U_x(N-1)$ from a *sufficient tight* scheme are replaced as follows $U_x \to U_x \otimes En(k+1)$. Revising of observable or measurement is another independent step. Assume, the N bit message is already encoded by entangled states $\Phi_x(N)$. Then before measuring, these states can disentangled by operator $Den(n)$, that produces n independent qubits, where $n \leq N$. The measurement becomes more simple because observable can be described by a set of states in which not all qubits, or maybe all of them are independent. The cost of modification is $U_x \to Den(n) \otimes U_x$. As result, the main revising of the dense coding *sufficient tight* schemes is

$$U_x \to Den(n) \otimes U_x \otimes En(k+1). \tag{33}$$

In the case of the GHZ channel (33) takes the form $U_x \to H_B C_{BC} \otimes U_x(2) \otimes C_{BC}$.

Both the entanglement and the disentanglement operators are useful for modification of the *sufficient tight* teleportation schemes for which there are some ways how to prepare the channel state Ω. As the N - particle channel allows transmitting perfectly only the $N-1$ particle entangled state of the form $\zeta(N-1) = \alpha|0\rangle^{\otimes(N-1)} + \beta|1\rangle^{\otimes(N-1)}$, then one of the idea of modification is disentanglement of the state to be teleported: $\zeta(N-1) \to (\alpha|0\rangle + \beta|1\rangle) \otimes |N-2\rangle$, where $N-2$ particles in state $|N-2\rangle$ can be entangled with two ancilla qubits, say in the EPR state, for preparing the quantum channel Ω. Combining with disentanglement operations it results in a collection of schemes, which based on one EPR pair and the Bell state measurement as one of the initial resource, however the measurement will involve not all particles to be teleported.

We illustrate the generalization of the one-bit teleportation protocol, considering for simplicity how to transmit two entangled qubits. It can be done, if an unknown state $\zeta = (\alpha|00\rangle + \beta|11\rangle)_{12}$ is entangled with an EPR pair of the form $\Omega = (|00\rangle + |11\rangle)_{AB}/\sqrt{2}$ as follows $C_{A2}|\zeta\rangle_{12} \otimes |\Omega\rangle_{AB}$. Then the joint measurement of the qubit 1 and 2 in basis $\pi_1^{\pm} \otimes |b\rangle_2$, $b = 0, 1$ projects the remainder qubits A and B onto the state to be equal to the unknown state up to unitary transformations. Note, here the non-Bell state measurement allows teleporting two entangled qubits by the 2 bits of classical information, however it requires the non-local operations.

5. Conclusions

We have studied quantum information tasks such as teleportation of entangled states and dense coding based on the N-particle quantum channel of the GHZ-class. In the multiqubit dense coding scheme we introduced the protocol how any N-bit message could be transmitted by manipulating of the $N-1$ qubits of the channel. This property arises from the fact, that a complete set of the N-particle entangled states of the GHZ form can be generated from one of them by a collection of the $N-1$ particle operators. It results in enhancement of the classical capacity by $N/N-1$ times due from entanglement. By contrast the Werner theorem, proved for a two-particle channel, resources of the dense coding cannot be used for teleportation, when $N > 2$ and visa versa. The multiparticle channel is suitable also for distributing of unknown states or telecloning. When the GHZ channel shared three parties, then one of them can transmit an unknown qubit in a mixed state and the others find in their hands perfect copies, that are classically correlated. The telecloning protocol could be useful as a pumping mechanism in the micromaser theory. A set of new schemes of teleportation and dense coding can be introduced, if two operations as entanglement and disentanglement are permitted. As result, by this way generalization of the one-bit teleportation is obtained, but it cannot be done locally.

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