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OPTIMAL TWO-PARTICLE ENTANGLEMENT BY UNIVERSAL QUANTUM PROCESSES

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Within the class of all possible universal (covariant) two-particle quantum processes in arbitrary dimensional Hilbert spaces those universal quantum processes are determined whose output states optimize the recently proposed entanglement measure of Vidal and Werner. It is demonstrated that these optimal entanglement processes belong to a one-parameter family of universal entanglement processes whose output states do not contain any separable components. It is shown that these optimal universal entanglement processes generate antisymmetric output states and, with the single exception of qubit systems, they preserve information about the initial input state.

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1. Introduction

One of the main driving forces in the rapidly developing field of quantum information processing is the question whether basic quantum phenomena such as interference and entanglement can be exploited for practical purposes. In this context it has been realized that the linear character of quantum theory may impose severe restrictions on the performance of elementary tasks of quantum information processing. As a consequence it is impossible to copy (or clone) an arbitrary quantum state perfectly [1].

In view of the significance of entangled states for many aspects of quantum information processing [2, 3, 4] the natural question arises whether similar restrictions also hold for quantum mechanical entanglement processes. Of particular interest are entanglement processes which entangle two quantum systems in an optimal way. Though many quantum mechanical processes are capable of entangling some input states of a quantum system with a known reference state of a second quantum system, it is not easy to achieve this goal for all possible input states. This basic difficulty can be realized already in the simple example of a quantum mechanical controlled-not (CNOT) operation, i. e. CNOT : $|\pm\rangle \otimes (|+\rangle + |-\rangle) \rightarrow |\mp\rangle \otimes |+\rangle + |\pm\rangle \otimes |-\rangle$. This CNOT operation entangles the orthogonal input states $|\pm\rangle$ of the first qubit with the second (control) qubit prepared in the reference state $(|+\rangle + |-\rangle)$. Obviously the two Bell states resulting from these input states are optimally entangled. However, due to its linearity this quantum process is incapable of entangling the first qubit with the second one for all possible input states. The input state $(|+\rangle + |-\rangle)$, for example, results in the factorizable output state $(|+\rangle + |-\rangle) \otimes (|+\rangle + |-\rangle)$. In view of this difficulty it is of particular interest to investigate universal entanglement processes which are able to entangle all possible input states of a quantum system with a second one in an optimal way. In this context entanglement can be considered as being optimal if the resulting two-particle output state does not contain any separable components.

Universal quantum processes act on all possible (typically pure) input states of a quantum system in a 'similar' way. Consequently, these processes do not specify a preferred direction in Hilbert space and thus reflect its 'natural' symmetry. Therefore, the restrictions imposed on these processes by the linear character of quantum theory are not only of practical interest but they also hint at fundamental limits of quantum theory. So far many properties of universal quantum processes have been analyzed for qubits [5, 6, 7]. For qubits one can show that there is only one universal entanglement process whose output states do not contain any separable components. Independent of the input states, this process always produces the anti-symmetric Bell state as the optimally entangled output state [8]. For many applications in quantum information processing, such as quantum error correction, universal quantum processes are of interest which do not only entangle different quantum systems in an optimal way but which also preserve information about the original input state and redistribute this information between the entangled quantum systems. Motivated by this need recently Bužek and Hillery have analyzed quantum processes which entangle two qubits and which also preserve information about the initial input state [6]. Though, in the case of qubits, both requirements are incompatible for universal quantum processes, universal optimal cloning processes manage to optimize both tasks simultaneously. However, the resulting output states always contain a separable two-qubit state. From these investigations on qubit systems one may be tempted to presume that a similar incompatibility between optimal universal entanglement processes and preservation of information about input states also holds in higher dimensional Hilbert spaces.

In this paper it is shown that, contrary to this tempting presumption, in Hilbert spaces of dimensions higher than two optimal universal entanglement processes are possible which simultaneously also preserve information about the initial input state. For this purpose a theoretical framework is developed within which all possible bipartite universal quantum processes can be described. For the sake of simplicity we restrict our discussion to the important special case that the dimensions of the Hilbert spaces of both quantum systems are equal. First of all, the class of all possible universal quantum processes is determined which is compatible with the linear character of quantum mechanics. Secondly, the particular subclass is determined which produces entangled two-particle output states which do not contain any separable components. It turns out that for Hilbert spaces with dimensions larger than two these particular universal entanglement processes form a one-parameter family. It is shown that the optimal universal quantum processes whose output states optimize the recently proposed entanglement measure of Vidal and Werner always belong to this family [9].

This paper is organized as follows: In Sec. 2 the basic symmetry (or covariance) property of universal quantum processes is discussed by starting from a simple example. Subsequently a general formalism is developed for describing all possible universal quantum processes in arbitrary dimensional Hilbert spaces. The consequences of covariance and of the linear character of universal quantum processes are implemented. In Sec. 3 all universal entanglement processes are determined whose output states do not contain any separable components. Subsequently the universal quantum processes are determined whose output states optimize the entanglement measure of Vidal and Werner [9]. Finally basic properties of the resulting optimally entangled output states are discussed.

2. Universal quantum processes

In this section the symmetry (or covariance) property of universal quantum processes is exemplified by considering two qubits. Based on this covariance property and on the requirement that any quantum process has to be linear with respect to all possible input states the general structure of universal (or covariant) quantum processes is discussed for the case of two arbitrary dimensional quantum systems of equal dimensions. Optimal universal quantum cloning processes and optimal universal entanglement processes are special cases thereof.

2.1. *Universal quantum processes and covariance – an example*

Let us consider the following quantum process as an introductory example:
Initially we prepare two distinguishable spin-1/2 quantum systems (qubits) in the state

$$\rho_1(\mathbf{m}) \equiv \rho_{in}(\mathbf{m}) \otimes \frac{1}{2}\mathbf{1}.$$

The pure input state $\rho_{in}(\mathbf{m}) = |\mathbf{m}\rangle\langle\mathbf{m}|$ of the first quantum system can be described by its Bloch vector $\mathbf{m}$. This Bloch vector can take an arbitrary position on the Poincare sphere. The second quantum system is in a completely unpolarized reference state which is assumed to be fixed once and for all. Selecting an arbitrary pure input state $\rho_{in}(\mathbf{m})$ we transfer the initial state $\rho_1(\mathbf{m})$ into the output state

$$\rho_1(\mathbf{m}) \to \rho_2(\mathbf{m}) = \frac{\mathbf{P}_J \rho_1(\mathbf{m}) \mathbf{P}_J}{\mathrm{Tr}[\mathbf{P}_J \rho_1(\mathbf{m}) \mathbf{P}_J]}. \tag{1}$$

Thereby the projection operator $\mathbf{P}_J = \sum_M |JM\rangle\langle JM|$ projects onto two-particle states with well defined total angular momentum J. This total angular momentum can assume the possible values $J = 1$ or $J = 0$ so that we can distinguish between two quantum processes. In a probabilistic way the transformation of Eq.(1) can be achieved by a measurement process with probability $\mathrm{Tr}[\mathbf{P}_J \rho_1(\mathbf{m}) \mathbf{P}_J]$. However, one may also think of realizing this transformation with a probability arbitrarily close to unity by some other means. Choosing the direction of polarization of the input state as the quantization axis the result of this quantum process is given either by

$$\begin{aligned} \rho_2(\mathbf{m}) &= p_1 |J=1\ M=1\rangle\langle J=1\ M=1| \\ &\quad + (1-p_1)|J=1\ M=0\rangle\langle J=1\ M=0| \end{aligned} \tag{2}$$

with $p_1 = 2/3$ or by

$$\rho_2(\mathbf{m}) = |J = 0\ M = 0\rangle\langle J = 0\ M = 0| \tag{3}$$

depending on whether $J = 1$ or $J = 0$. Both quantum processes are universal in the sense that all input states are treated in a 'similar' way. In particular, this implies that the probabilities entering Eq.(2) are independent of the input state $|\mathbf{m}\rangle$. The only direction the output state depends on is the one of the input state. Thus these quantum processes are symmetric with respect to unitary transformations U which transform an arbitrary pure one-particle input state, say $|\mathbf{m}_0\rangle$, into some other pure one-particle input state, say $|\mathbf{m}\rangle \equiv U(\mathbf{m})|\mathbf{m}_0\rangle$. This unitary symmetry or covariance of such a universal quantum process is characterized by the relation

$$\rho_2(\mathbf{m}) = U(\mathbf{m}) \otimes U(\mathbf{m})\rho_2(\mathbf{m}_0)U^\dagger(\mathbf{m}) \otimes U^\dagger(\mathbf{m}) \tag{4}$$

(compare with Fig.1). Thus the possible output states of a universal quantum process constitute a two-particle representation of the group of unitary one-particle transformations. The covariance condition of Eq.(4) already describes how these particular quantum processes can be realized. A covariant quantum process is initialized by preparing the two-particle quantum system in a particular state, say $\rho_2(\mathbf{m}_0)$, which is associated with the particular pure input state $|\mathbf{m}_0\rangle$. The output state of any other input state, say $|\mathbf{m}\rangle = U(\mathbf{m})|\mathbf{m}_0\rangle$, is obtained by applying to both particles the unitary two-particle operation $U(\mathbf{m}) \otimes U(\mathbf{m})$.

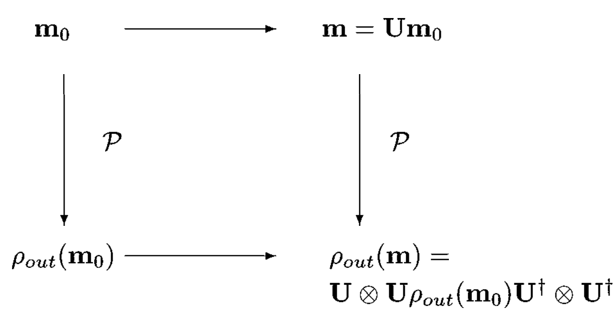

Fig. 1. Pictorial representation of the symmetry (covariance) condition which characterizes universal quantum processes.

Universal quantum processes in which the step of Eq.(1) can be implemented with a probability of unity have been investigated in the context of copying (cloning) quantum states. In particular, it has been demonstrated that optimal quantum cloning can be achieved always by a universal quantum process. Furthermore, in the case of two qubits the maximum probability with which an optimal universal quantum cloning process is successful is given by 2/3 [5]. This latter probability is identical with the probability p_1 appearing in Eq.(2). Thus, provided the process of Eq.(1) with $J = 1$ is implemented with a probability of unity this process copies an arbitrary input state in an optimal way. However, if we consider the process projecting onto states with $J = 0$, we end up in the anti-symmetric Bell state formed by both qubits. This output state is independent of the input state which we choose. As a Bell state is maximally entangled this latter type of process is an example of a universal optimal entanglement process.

Copying quantum states and preparing entangled quantum states are elementary tasks of quantum information processing. Thereby universal quantum processes fulfilling Eq.(4) which exhibit the same symmetry as the set of all possible pure one-particle input states are of special interest. Though much is already known about universal quantum cloning processes almost nothing is known about universal quantum processes which yield optimally entangled quantum states, in particular in arbitrary dimensional Hilbert spaces. The main questions which will be addressed in the following are: Which entangled quantum states result from universal quantum processes which maximize entanglement? Which limitations are imposed on the structure of these states by the universality and linearity of these quantum processes? How do the properties of the resulting optimally entangled quantum states depend on the dimensionality of the Hilbert spaces involved?

2.2. *General structure of universal quantum processes involving two quantum systems of equal dimensions*

Let us consider the most general universal quantum process of the form

$$\mathcal{P} : \rho_{in}(\mathbf{m}) \otimes \rho_{ref} \rightarrow \rho_{out}(\mathbf{m}). \tag{5}$$

In our previous example the fixed reference state ρ_{ref} was maximally mixed. In the present case we leave its form unspecified. The density operator of the pure input state is denoted $\rho_{in}(\mathbf{m}) \equiv |\mathbf{m}\rangle\langle\mathbf{m}|$. For the sake of simplicity let us assume that the dimensions of the Hilbert spaces for both quantum systems are equal and of magnitude $D \geq 2$. In order to classify all possible universal quantum processes of the form of Eq.(5) we have to determine the most general form of output states.

The density operator of an arbitrary input state of a D dimensional quantum system can always be represented in terms of the generators $\mathbf{A}_{ij}$ $(i, j = 1, ..., D)$ of the group SU_D, i.e.

$$\rho_{in}(\mathbf{m}) = \frac{1}{D}(\mathbf{1} + m_{ij}\mathbf{A}_{ij}). \tag{6}$$

(We use the Einstein summation convention in which one has to sum over all indices $i, j \in \{1, .., D\}$ which appear in an expression twice.) A representation of these generators is given by the $D \times D$ matrices

$$(\mathbf{A}_{ij})^{(kl)} = \delta_{ik}\delta_{jl} - \frac{1}{D}\delta_{ij}\delta_{kl}. \tag{7}$$

These matrices are not hermitian but they fulfill the relation $\mathbf{A}_{ij}^{\dagger} = \mathbf{A}_{ji}$. Due to the constraint $\sum_{i=1}^{D} \mathbf{A}_{ii} = 0$ only $(D^2 - 1)$ of them are linearly independent so that we may choose $m_{DD} = 0$ in Eq.(6). For $D = 2$ these matrices reduce to the well known spherical components of the Pauli spin matrices, i.e. $2\mathbf{A}_{11} = \sigma_z$, $2\mathbf{A}_{12} = \sigma_x + i\sigma_y$ and $2\mathbf{A}_{21} = \sigma_x - i\sigma_y$. Furthermore, $\rho_{in}(\mathbf{m}) = \rho_{in}(\mathbf{m})^{\dagger}$ implies the relations $[m_{ij}]^* = m_{ji}$ so that Eq.(6) involves $(D^2 - 1)$ real-valued and linearly independent parameters which form the components of a generalized Bloch vector. For $i < j$ one may choose the real and imaginary parts of m_{ij} as linearly independent parameters and for $1 \leq i \leq D-1$ the diagonal elements m_{ii}. The non-negativity of the density operator $\rho_{in}(\mathbf{m})$ imposes further restrictions on these parameters [10]. However, their explicit form is not important for our subsequent discussion in which we are interested in pure input states only. Without loss of generality, the covariance condition (4) implies that we can restrict ourselves to a pure input state which coincides with one of the basis vectors, say $|1\rangle$, of the D-dimensional Hilbert space. The associated density operator of the input state is given by

$$\rho_{in}(\mathbf{m}_0 = D\mathbf{A}_{11}) = \frac{1}{D}(\mathbf{1} + D\mathbf{A}_{11}) \equiv |1\rangle\langle 1| \tag{8}$$

with $m_{ij} = \delta_{i1}\delta_{j1}D$. According to the covariance condition (4) any output state can be obtained from the associated output state $\rho_{out}(\mathbf{m}_0 = D\mathbf{A}_{11})$ by a local, unitary two-particle transformation.

In terms of the generators of Eq.(7) the most general two-particle output state is represented by a density operator of the form

$$\begin{aligned}\rho_{out}(\mathbf{m}) \quad = \quad & \frac{1}{D^2}\mathbf{1}\otimes\mathbf{1} + \alpha^{(1)}_{ij}(\mathbf{m})\mathbf{A}_{ij}\otimes\mathbf{1} + \\ & \alpha^{(2)}_{ij}(\mathbf{m})\mathbf{1}\otimes\mathbf{A}_{ij} + K_{ijkl}(\mathbf{m})\mathbf{A}_{ij}\otimes\mathbf{A}_{kl}.\end{aligned} \tag{9}$$

In order to implement the covariance condition (4) and the linearity requirement of quantum processes it is useful to separate the last term of Eq.(9) into terms which are invariant and into terms which transform as the generators $\mathbf{A}_{ij}$ under arbitrary unitary transformations of the form $U\otimes U$. For this purpose, let us start from the commutation relations of SU_D, namely

$$[\mathbf{A}_{ij}, \mathbf{A}_{mn}] = \mathbf{A}_{ab}(\delta_{jm}\delta_{ai}\delta_{bn} - \delta_{in}\delta_{am}\delta_{bj}). \tag{10}$$

These relations imply that the tensor products $\mathbf{A}_{ji}\otimes\mathbf{A}_{sj}$ transform under arbitrary transformations of the form $U\otimes U$ in the same way as $\mathbf{A}_{si}$ transforms under transformation of the form U. Furthermore, the tensor product $\mathbf{A}_{ij}\otimes\mathbf{A}_{ji}$ is an invariant under arbitrary unitary transformations of the form $U\otimes U$. However, note that the combination $\mathbf{A}_{ij}\otimes\mathbf{A}_{sj}$, for example, does not transform analogous to $\mathbf{A}_{si}$. Using these elementary transformation properties, the covariance condition (4), and the fact that any quantum operation has to be linear with respect to its input states the density operator of the two-particle output state has to be of the form

$$\begin{aligned}\rho_{out}(\mathbf{m}) \quad = \quad & \frac{1}{D^2}\mathbf{1}\otimes\mathbf{1} + \alpha^{(1)}_{ij}(\mathbf{m})\mathbf{A}_{ij}\otimes\mathbf{1} + \\ & +\alpha^{(2)}_{ij}(\mathbf{m})\mathbf{1}\otimes\mathbf{A}_{ij} + C\mathbf{A}_{ij}\otimes\mathbf{A}_{ji} + \\ & \beta_{il}(\mathbf{m})\mathbf{A}_{ij}\otimes\mathbf{A}_{jl} + \beta_{il}(\mathbf{m})^*\mathbf{A}_{ji}\otimes\mathbf{A}_{lj}\end{aligned} \tag{11}$$

with

$$\alpha^{(1,2)}_{ij} = \alpha^{(1,2)}m_{ij}, \ \beta_{ij} = \beta m_{ij} \tag{12}$$

and with $C \in \mathbf{R}$ being independent of $\mathbf{m}$.

So far the output state of Eq.(11) represents the most general hermitian operator which depends linearly on the input state $\rho_{in}(\mathbf{m})$ and which fulfills the covariance condition (4). Accordingly, a particular universal quantum process is characterized by the set of real-valued parameters C, $\alpha^{(1)}$, $\alpha^{(2)}$ and by the complex valued parameter β. We still have to solve the more difficult task to restrict the range of these parameters in such a way that $\rho_{out}(\mathbf{m})$ of Eq.(11) represents a non-negative operator. In order to determine this fundamental range of these parameters we have to investigate the possible eigenvalues of the density operator $\rho_{out}(\mathbf{m})$ of Eq.(11). Due to the covariance condition (4) we may restrict this investigation to a particular pure input state, say $\rho_{in}(\mathbf{m}_0 = D\mathbf{A}_{11}) = |1\rangle\langle 1|$. Using the matrix representations of Eq.(7) it turns out that the corresponding output state can be represented by a direct sum of density operators according to

$$\rho_{out}(\mathbf{m}_0 \equiv D\mathbf{A}_{11}) = \sum_{i=1}^{4} \oplus p_i \rho_i \tag{13}$$

with the partial density operators

$$\begin{aligned}
\rho_1 &= |11\rangle\langle 11|, \\
\rho_2 &= \sum_{j=2}^{D} \{|1j\rangle\langle 1j|(\frac{1}{2(D-1)} + \frac{(\alpha^{(1)}-\alpha^{(2)})m_{11}}{2p_2}) + \\
&\quad |j1\rangle\langle j1|(\frac{1}{2(D-1)} + \frac{(\alpha^{(2)}-\alpha^{(1)})m_{11}}{2p_2}) + \\
&\quad |1j\rangle\langle j1|\frac{C+\beta m_{11}}{p_2} + |j1\rangle\langle 1j|\frac{C+\beta^* m_{11}}{p_2}\}, \\
\rho_3 &= \frac{1}{(D-1)} \sum_{j=2}^{D} |jj\rangle\langle jj|, \\
\rho_4 &= \sum_{2=i<j}^{D} \{|ij\rangle\langle ij| \frac{1}{(D-1)(D-2)} + \\
&\quad |ji\rangle\langle ji| \frac{1}{(D-1)(D-2)} + \\
&\quad |ij\rangle\langle ji|\frac{C}{p_4} + |ji\rangle\langle ij|\frac{C}{p_4}\}.
\end{aligned} \tag{14}$$

These partial density operators are normalized so that $\mathrm{Tr}(\rho_i) = 1$ for $i = 1, ..., 4$. The corresponding partial probabilities entering Eq.(13) are given by

$$\begin{aligned}
p_1 &= \frac{1}{D^2} + (\alpha^{(1)}+\alpha^{(2)})m_{11}(1-\frac{1}{D}) + C(1-\frac{1}{D}) + \\
&\quad (\beta+\beta^*)m_{11}(1-\frac{1}{D})^2, \\
p_2 &= (D-1)\{\frac{2}{D^2} + (\alpha^{(1)}+\alpha^{(2)})m_{11}(1-\frac{2}{D}) - \\
&\quad \frac{2C}{D} - 2(\beta+\beta^*)m_{11}(1-\frac{1}{D})\frac{1}{D}\},
\end{aligned}$$

$$\begin{aligned}
p_3 &= (D-1)\{\frac{1}{D^2} - \frac{\alpha^{(1)}m_{11}}{D} - \frac{\alpha^{(2)}m_{11}}{D} + \\
&\quad C(1-\frac{1}{D}) + (\beta+\beta^*)m_{11}\frac{1}{D^2}\},\\
p_4 &= (D-1)(D-2)\{\frac{1}{D^2} - \frac{\alpha^{(1)}m_{11}}{D} - \frac{\alpha^{(2)}m_{11}}{D} - \\
&\quad \frac{C}{D} + (\beta+\beta^*)m_{11}\frac{1}{D^2}\}.
\end{aligned} \tag{15}$$

The normalization of the density operator, i.e. $\mathrm{Tr}[\rho_{out}(\mathbf{m})] = 1$, implies

$$p_1 + p_2 + p_3 + p_4 = 1. \tag{16}$$

From Eqs.(14) and (15) one obtains the eigenvalues of $\rho_{out}(\mathbf{m}_0 = D\mathbf{A}_{11})$, namely

$$\begin{aligned}
\lambda_1 &= p_1,\\
\lambda_{2\pm} &= \frac{p_2}{2(D-1)} \pm \sqrt{(\frac{(\alpha^{(1)}-\alpha^{(2)})m_{11}}{2})^2 + \mid C + m_{11}\beta \mid^2},\\
\lambda_3 &= \frac{p_3}{(D-1)},\\
\lambda_{4\pm} &= \frac{p_4}{(D-1)(D-2)} \pm \mid C \mid .
\end{aligned} \tag{17}$$

Therefore the density operator of Eq.(13) is non-negative only if all probabilities p_i and all eigenvalues λ_i of Eqs.(15) and (17) are non-negative and fulfill Eq.(16). For $\alpha^{(1)} = \alpha^{(2)}$ and $\beta = \beta^*$, for example, these conditions on (p_2, p_3, p_4) form a tetrahedron (compare with Fig. 2). Each point in this convex set defines a unique universal quantum process whose possible output states can be obtained from Eq.(13) with the help of the covariance condition (4). The universal quantum cloning process, for example, is represented by point B in this figure and it is characterized by the particular universal process which maximizes p_1. Note that it is immediately obvious from Fig. 2 that perfect quantum cloning is impossible with a universal quantum process as $p_1 = 1 - p_2 - p_3 - p_4 \leq 2/(D+1) < 1$ for $D \geq 2$.

Finally, it should be mentioned that for dimensions $D \geq 3$ one may choose the probabilities (p_1, p_3, p_4) or (p_2, p_3, p_4), for example, as independent coordinates instead of the three independent real-valued parameters $((\alpha^{(1)} + \alpha^{(2)}), C, (\beta + \beta^*))$. Inverting Eqs.(15) and using Eq.(16) one obtains the relation between these different coordinates, namely

$$\begin{aligned}
\beta + \beta^* &= -\frac{1}{D(D-1)} + \frac{p_4}{(D-1)(D-2)} + \frac{p_1}{(D-1)},\\
\alpha^{(1)} + \alpha^{(2)} &= \frac{(D-2)}{D^2(D-1)} - \frac{p_4}{D(D-1)} + \frac{p_1}{D(D-1)} - \frac{p_3}{D(D-1)},\\
C &= \frac{p_3}{(D-1)} - \frac{p_4}{(D-1)(D-2)}.
\end{aligned} \tag{18}$$

In order to identify a particular universal quantum process uniquely in addition to these three probabilities one also has to specify the remaining two independent parameters, namely $(\alpha^{(1)} - \alpha^{(2)})$ and $(\beta - \beta^*)$.

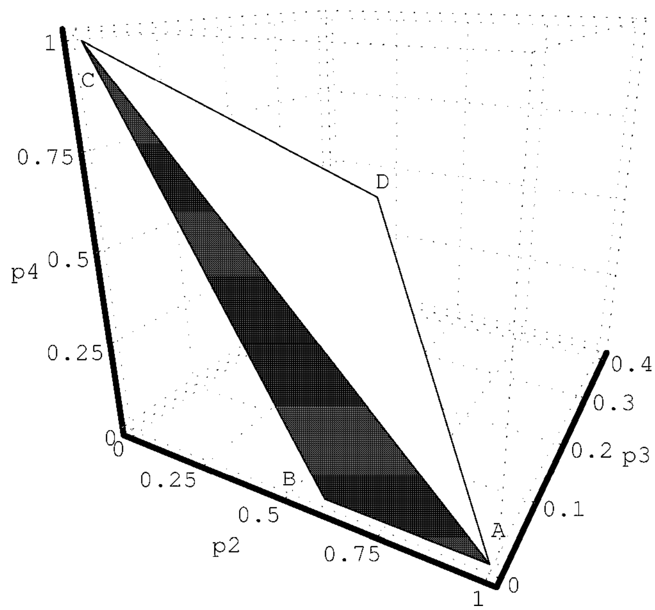

Fig. 2. Convex set of points (p_2, p_3, p_4) characterizing all possible universal quantum processes for $\alpha^{(1)} = \alpha^{(2)}$, $\beta = \beta^*$ and $D = 4$.

3. Universal entanglement processes

In this section it is shown that there is a unique one-parameter family of universal entanglement processes whose resulting output states do not contain any separable components. These processes produce output states which are anti-symmetric with respect to particle exchange. It is demonstrated that the universal quantum processes whose output states maximize the recently proposed entanglement measure of Vidal and Werner are always members of this one-parameter family [9]. Basic properties of the output states resulting from these optimal universal entanglement processes are discussed and it is investigated to which degree these output states preserve information about the input state.

3.1. *Universal entanglement processes yielding output states without separable components*

Is it possible to entangle two quantum systems in such a way by a universal quantum process that the resulting output states do not contain any separable components?

As discussed by Lewenstein and Sanpera one can decompose any quantum state ρ of a two-particle system into a separable part, say ρ_{sep}, and an inseparable contribution ρ_{insep}, i.e. $\rho = \lambda\rho_{sep} + (1-\lambda)\rho_{insep}$ with $0 \leq \lambda \leq 1$ [11]. Thereby a separable state is a convex sum of product states of the form $\rho_A \otimes \rho_B$ where ρ_A and ρ_B refer to quantum systems A and B respectively. Though in general this decomposition is not unique the optimal decomposition with maximal λ is unique. Thus, with respect to entanglement those quantum states are of particular interest whose maximum possible value of λ equals zero in any such decomposition [12, 13].

In order to determine the parameters for the universal quantum processes which produce this latter class of entangled states let us start from the output state $\rho_{out}(\mathbf{m}_0 = D\mathbf{A}_{11})$ of Eq.(13). A necessary requirement for this state belonging to this class is the absence of any admixtures of separable states of the form $|jj\rangle\langle jj|$ for $j = 1, ..., D$. Thus, necessarily a universal quantum process producing entangled states without any separable components has to be characterized by the parameters

$$p_1 = 0, \; p_3 = 0. \tag{19}$$

It will be demonstrated by the subsequent arguments that this choice of parameters is also sufficient for the generation of entangled output states without separable components. For this purpose it has to be proven that for any separable two-particle state $|\psi\rangle = |\varphi\rangle \otimes |\chi\rangle$ and for any positive value of $\lambda > 0$ the state

$$\rho' = \rho_{out}(\mathbf{m}_0 = D\mathbf{A}_{11})^{(ent)} - \lambda|\psi\rangle\langle\psi| \tag{20}$$

is negative definite. Thereby the state $\rho_{out}(\mathbf{m}_0 = D\mathbf{A}_{11})^{(ent)}$ fulfills conditions (19). Consequently, the covariance condition (4) and the non-negativity property of density operators implies that any arbitrary output state $\rho_{out}(\mathbf{m})^{(ent)}$ fulfilling Eqs.(19) cannot contain any separable components.

For the proof of this latter statement we start from conditions (19) and Eqs.(15), (17) and (18). According to Eqs.(17) and (18) the condition $p_3 = 0$ implies $\lambda_{4-} = 0$. Furthermore, from the non-negativity of λ_{2-} of Eq.(17) and from Eqs. (18) and (19) we obtain the relations

$$\begin{aligned} \alpha^{(1)} &= \alpha^{(2)}, \\ \beta &= \beta^*, \\ \rho_2^{(ent)} &= \frac{1}{2(D-1)} \sum_{j=2}^{D} \{|1j\rangle\langle 1j| + |j1\rangle\langle j1| - \\ & |1j\rangle\langle j1| - |j1\rangle\langle 1j|\}, \\ \rho_4^{(ent)} &= \frac{1}{(D-1)(D-2)} \sum_{2=i<j}^{D} \{|ij\rangle\langle ij| + |ji\rangle\langle ji| - \\ & |ij\rangle\langle ji| - |ji\rangle\langle ij|\}. \end{aligned} \tag{21}$$

Thus, the parameters of Eqs.(19) imply that the resulting output state

$$\rho_{out}^{(ent)}(\mathbf{m}_0 = D\mathbf{A}_{11}) = (1-p_4)\rho_2^{(ent)} \oplus p_4\rho_4^{(ent)} \tag{22}$$

is a convex sum of pure two-particle quantum states which are anti-symmetric with respect to permutations of both quantum systems, i.e. a convex sum of two-particle Slater determinants. Let us consider now the state ρ' of Eq.(20). For an arbitrary state $|\psi\rangle = |\varphi\rangle \otimes |\chi\rangle$ we can always choose a unitary transformation U in such a way that $\langle 1|U|\varphi\rangle$ and $\langle 1|U|\chi\rangle$ are both non-zero. This unitary transformation may be interpreted passively as a change of basis in the one-particle Hilbert spaces. Applying the same unitary transformation to state $\rho_{out}^{(ent)}(\mathbf{m}_0 = D\mathbf{A}_{11})$ a convex sum of anti-symmetric two-particle states is produced so that $\langle 11|U \otimes U\rho_{out}^{(ent)}U^\dagger \otimes U^\dagger|11\rangle = 0$. Thus, assuming the existence of a state $|\psi\rangle = |\varphi\rangle \otimes |\chi\rangle$ and a probability $\lambda > 0$ implies that for this particular unitary transformation U the diagonal density matrix element $\langle 11|U\otimes U\rho' U^\dagger \otimes U^\dagger|11\rangle = 0 - \lambda\langle 1|U|\varphi\rangle\langle 1|U|\chi\rangle$ is negative. Therefore ρ' is negative definite for any choice of the states $|\varphi\rangle$ and $|\chi\rangle$ and for any $\lambda > 0$. Correspondingly a non-zero value of λ is not possible in Eq.(20). So we conclude that the two-particle state of Eq.(22) does not contain any separable component. By covariance the same property applies to all possible output states. This completes our proof.

3.2. *Optimal universal entanglement processes*

Which universal quantum processes optimize entanglement according to the recently proposed entanglement measure of Vidal and Werner [9]?

In order to answer this question let us, first of all, briefly summarize basic aspects of this entanglement measure. According to Vidal and Werner the negativity $N(\rho)$ of any two-particle density operator ρ, i.e.

$$N(\rho) = |\sum_i \mu_i|, \tag{23}$$

is a measure of entanglement [9]. Thereby μ_i are the negative eigenvalues of the partial transpose ρ^T [14] of ρ. This entanglement measure is monotonic under local operations and classical communication and it is convex, i.e.

$$N(\sum_i p_i\rho_i) \leq \sum_i p_i N(\rho_i) \tag{24}$$

for density operators ρ_i and for $p_i \geq 0$ with $\sum_i p_i = 1$ [14].

The convexity of this measure can be used to determine the universal entanglement processes which yield maximally entangled output states. For this purpose it is sufficient to consider the particular output state of the most general bipartite universal quantum process given by Eq.(13). The convexity of the entanglement measure $N(\rho)$ implies the inequality

$$\begin{aligned} &N(\rho_{out}(\mathbf{m}_0)) \leq (p_1+p_3)N(\frac{p_1\rho_1+p_2\rho_3}{p_1+p_3})+ \\ &(p_2+p_4)N(\frac{p_2\rho_2+p_4\rho_4}{p_4+p_4}) = (p_2+p_4)N(\frac{p_2\rho_2+p_4\rho_4}{p_4+p_4}) \leq \\ &p_2N(\rho_2)+p_4N(\rho_4) \equiv \sqrt{D-1}|C+\beta m_{11}| + (D-2)|C|. \end{aligned} \tag{25}$$

The first equality involved in (25) follows from the fact that ρ_1 and ρ_3 are diagonal matrices (compare with Eq.(14)) and thus the negativity of any convex sum of these density matrices vanishes. The second inequality involved in (25) follows from a second application of the convexity of the entanglement measure $N(\rho)$. The last equality in (25) follows from a straightforward evaluation of the entanglement measures of ρ_2 and of ρ_4 on the basis of Eq.(14). From Eqs. (17) and (18) we obtain the additional upper bounds

$$\begin{aligned} |C| &\leq \frac{p_4}{(D-1)(D-2)}, \\ |C+\beta m_{11}|^2 &\leq (\frac{p_2}{2(D-1)})^2 - (\frac{(\alpha^1-\alpha^2)D}{2})^2. \end{aligned} \tag{26}$$

Inserting these latter inequalities into (25) we obtain the relation

$$N(\rho_{out}(\mathbf{m}_0)) \leq \frac{p_2}{2\sqrt{D-1}} + \frac{p_4}{(D-1)} \tag{27}$$

with $\sum_{i=1}^4 p_i = 1$. For arbitrary values of p_4 the right hand side of inequality (27) is maximal for $p_1 = p_3 = 0$. Thus we obtain the final inequality

$$N(\rho_{out}(\mathbf{m}_0)) \leq \frac{1}{2\sqrt{D-1}} + p_4(\frac{1}{D-1} - \frac{1}{2\sqrt{D-1}}) \tag{28}$$

with $0 \leq p_4 \leq 1$. For dimensions $D < 5$ the right hand side of inequality (28) is maximal for $p_4 = 0$ which is equivalent to $p_2 = 1$. Therefore, according to Eq.(14), in this case the universal quantum process with $p_2 = 1$ yields optimally entangled output states which saturate the upper bound of inequality (28). For dimensions $D > 5$ the maximum value of the right hand side of inequality (28) is achieved for $p_4 = 1$. Thus, in this latter case the universal quantum process with $p_4 = 1$ yields optimally entangled output states saturating the upper bound of inequality (28). The case of $D = 5$ is special in the sense that universal quantum processes with arbitrary values $0 \leq p_4 \leq 1$ and with $p_2 = 1 - p_4$ yield optimally entangled output states.

Thus, the optimal universal entanglement processes whose output states maximize the entanglement measure of Vidal and Werner always fulfill the condition $p_1 = p_3 = 0$ for arbitrary dimensions D [9]. Thus, they are always members of the one-parameter family of entanglement processes which do not yield any separable components and which were discussed in the previous subsection. In five dimensional one-particle Hilbert spaces optimal universal entanglement processes are special in the sense that they coincide with this previously discussed one-parameter family of entanglement processes.

3.3. *Basic properties of the resulting entangled output states*

The parameters

$$0 \leq p_4 \leq 1,\ p_1 = 0,\ p_3 = 0,\ \alpha^{(1)} = \alpha^{(2)},\ \beta = \beta^* \tag{29}$$

characterize all possible universal quantum processes which produce entangled two-particle output states which do not contain any separable components. One particular process within this one-parameter family of universal entanglement processes produces optimally entangled output states. For $D < 5$ this optimal entanglement process is characterized by the additional condition $p_2 = 1$ and for $D > 5$ it is characterized by the additional requirement $p_4 = 1$. The case $D = 5$ is special in the sense that all universal entanglement processes of Eq.(29) are optimal entanglement processes. The output states of the one-parameter family of universal quantum processes of Eq.(29) are statistical mixtures of anti-symmetric states. Explicitly they are given by Eq.(14) and by applying the covariance condition (4). In addition, these state also exhibit other noteworthy properties which will be discussed in the following.

The partial transpose of the output state $\rho_{out}^{(ent)}(\mathbf{m}_0 = D\mathbf{A}_{11})$ of Eq.(22) has always a negative eigenvalue of magnitude

$$\Lambda = -\frac{p_4}{2(D-1)} - \{\frac{p_4^2}{(D-1)^2} + \frac{(1-p_4)^2}{D-1}\}^{1/2}. \tag{30}$$

Therefore, by covariance the one-parameter family of universal entanglement processes of Eq.(29) produces free entangled states [15].

Due to covariance all output states resulting from the same universal optimal entanglement process have the same von Neumann entropy of magnitude

$$S(p_4) = p_4 \ln\frac{(D-1)(D-2)}{2p_4} + (1-p_4)\ln\frac{(D-1)}{1-p_4}. \tag{31}$$

Thus, for $D > 4$ the universal entanglement process with $p_4 = 0$ produces output states with the smallest possible von Neumann entropy, namely

$$S_{min} \equiv S(p_4 = 0) = \ln(D-1). \tag{32}$$

For $D < 4$ this process of minimal von Neumann entropy is characterized by $p_4 = 1$ and the corresponding minimal entropy is given by

$$S_{min} \equiv S(p_4 = 1) = \ln\frac{(D-1)(D-2)}{2}. \tag{33}$$

For $D = 4$ both processes, i.e. $p_4 = 0$ and $p_4 = 1$, yield the same von Neumann entropy for the output states. As apparent from Fig. 3, this possibility of a 'coexistence' of two universal entanglement processes with the same von Neumann entropy resembles some of the signatures of a second order phase transition. Within the one-parameter family of universal entanglement processes of Eq.(29) the process characterized by $p_4 = (D-2)/D$ (or equivalently $C =$

$-1/[D(D-1)]$) gives rise to output states with the largest possible value of the von Neumann entropy, namely

$$S_{max} \equiv S(p_4 = \frac{(D-2)}{D}) = \ln\frac{D(D-1)}{2}. \tag{34}$$

Thus this process generates an output state which is a maximal mixture of all possible $(D-1)(D-2)/2$ anti-symmetric two-particle states.

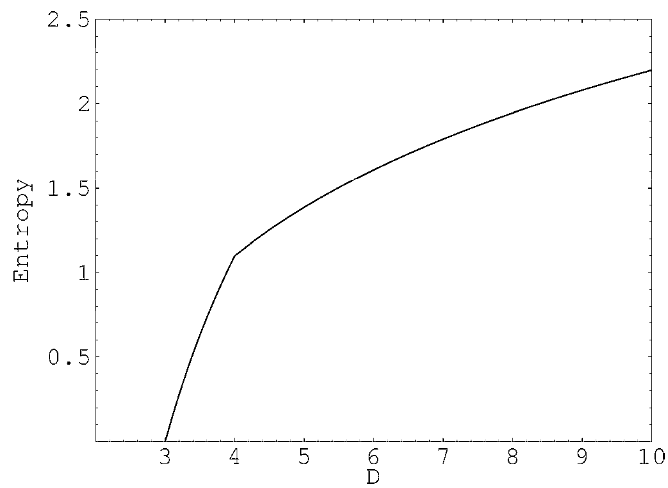

Fig. 3. Minimal values of the von Neumann entropy of optimal universal entanglement processes (compare with Eqs.(32) and (33)) as a function of D.

The index of correlation of the possible output states is defined by

$$IC(\rho) = S(R_1(\mathbf{m})) + S(R_2(\mathbf{m})) - S(\rho_{out}(\mathbf{m})) \tag{35}$$

with the reduced density operators of the first and second quantum system

$$R_1(\mathbf{m}) \equiv \mathrm{Tr}_2\{\rho_{out}(\mathbf{m})\},\ R_2(\mathbf{m}) \equiv \mathrm{Tr}_1\{\rho_{out}(\mathbf{m})\}. \tag{36}$$

This index of correlation or mutual entropy serves as a measure for the classical and quantum correlations between both quantum systems [16]. Due to the covariance condition (4) it is a property of a particular universal quantum process and is independent of the input state. For the one-parameter family of universal entanglement processes of Eq.(29) the index of correlation is given by

$$IC(p_4) = \ln\frac{4}{1+p_4} + p_4\ln\frac{2p_4(D-1)}{(1+p_4)(D-2)}. \tag{37}$$

From this relation it is apparent that $IC(p_4)$ has a local minimum for $p_4 = (D-2)/D$. Thus, the entanglement process with the largest possible von Neumann entropy produces output states with the smallest possible mutual entropy. Furthermore, the output states of the entanglement process with $p_4 = 0$ have the largest possible index of correlation, i.e. $IC(p_4 = 0) = 2\ln 2$. It is remarkable that this latter index of correlation is independent of the dimension of the Hilbert spaces D and that this value is equal to the mutual entropy of a Bell state.

It is also of interest to which extent the entangled output states resulting from the one-parameter family of universal entanglement processes of Eq.(29) preserve information about the initial pure input state $\rho_{in}(\mathbf{m})$. This information about the input state is characterized by the generalized Bloch vector $\mathbf{m}$. In the output state of Eq.(11) this information is contained in the terms proportional to the parameters $\alpha^{(1)}$, $\alpha^{(2)}$ and β. The parameters $\alpha^{(1)}$ and $\alpha^{(2)}$ characterize the information about the initial pure input state which is still contained in the two-particle output state in each subsystem separately, i.e. in the reduced states

$$R_1(\mathbf{m}) = \frac{\mathbf{1}}{D} + D\alpha^{(1)} m_{ij}\mathbf{A}_{ij}, \; R_2(\mathbf{m}) = \frac{\mathbf{1}}{D} + D\alpha^{(2)} m_{ij}\mathbf{A}_{ij} \tag{38}$$

of the first and second quantum system. The parameter β characterizes the information about the input state which is distributed over both quantum systems. This latter property is apparent from the fact that this parameter appears in Eq.(11) with tensor products of the form $\mathbf{A}_{ij} \otimes \mathbf{A}_{jl}$ and $\mathbf{A}_{ji} \otimes \mathbf{A}_{lj}$. According to Eqs.(18) and (29) for a given value of p_4 (with $p_1 = p_3 = 0$) these characteristic quantities are given by

$$\begin{aligned} \alpha^{(1)} + \alpha^{(2)} &= \frac{(D-2)}{D^2(D-1)} - \frac{p_4}{D(D-1)}, \\ \beta + \beta^* &= -\frac{1}{D(D-1)} + \frac{p_4}{(D-2)(D-1)}. \end{aligned} \tag{39}$$

Thus, the universal entanglement process with $p_4 = 0$ yields the maximal possible value for $\alpha^{(1)} \equiv \alpha^{(2)}$, namely

$$\alpha^{(1)}_{max} = (D-2)/[2D^2(D-1)] \tag{40}$$

and preserves the maximum amount of information about the initial state in each subsystem separately. It is instructive to compare this maximum value for $\alpha^{(1)}_{max}$ with the corresponding maximal value achievable by an optimal quantum cloning process. This latter optimal value is given by $\alpha^{(1)}_{clone} = (D-2)/[2D^2(D-1)] + 1/[D(D-1)(D+1)] \equiv (D+2)/[2D^2(D+1)]$ [5]. Thus, for $D > 2$ $\alpha^{(1)}_{clone}$ and $\alpha^{(1)}_{max}$ differ by terms of relative magnitude $O(1/D)$ so that their difference tends to zero with increasing dimension D of the one-particle Hilbert spaces. This demonstrates that for $D \gg 2$ a universal entanglement process with $p_4 = 0$ preserves almost as much information about the initial quantum state as an optimal universal cloning process (compare with Fig. 4).

Within the one-parameter family of Eq.(29) the universal entanglement process with $p_4 = (D-2)/D$ yields $\alpha^{(1)} = \alpha^{(2)} = \beta = 0$ so that all information about the orientation of the initial quantum state $\rho_{in}(\mathbf{m})$ is lost. The resulting output state is independent of the input state and is a scalar with respect to unitary transformations of the form $U \otimes U$ and with respect to permutations between both particles. This particular process is the only one within the one-parameter family of Eq.(29) which fulfills the additional requirement $R_1(\mathbf{m}) = R_2(\mathbf{m}) = \mathbf{1}/D$.

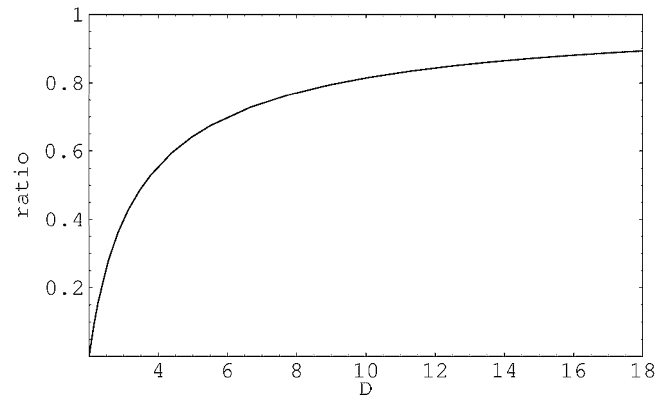

Fig. 4. Dimensional dependence of the ratio between $\alpha^{(1)}_{max}$ as defined by Eq.(40) and the corresponding value $\alpha^{(1)}_{clone}$ characterizing the optimal universal cloning process. It is for $D = 2$ only that in the optimal universal entanglement process all information about any input state is lost.

Though this property is characteristic for all Bell states it does not hold for the output states which are generated by the optimal universal entanglement processes discussed in Sec. 3.2.

As discussed in Sec. 3.2 optimal universal entanglement processes are characterized by $p_2 = 1$ for $D < 5$ and by $p_4 = 1$ for $D > 5$. The corresponding dimensional dependence of the entanglement measure $N(\rho)$ of these optimally entangled output states is depicted in Fig. 5. It is apparent that for $D = 5$ this entanglement measure is continuous but not differentiable. At this particular dimension the optimal universal entanglement processes discussed in Sec. 3.2 coincide with the one-parameter family of entanglement processes of Sec. 3.1 which yield entangled output states without any separable components.

3.4. *Examples*

In order to exemplify basic properties of entangled output states resulting from the one-parameter family of universal entanglement processes of Eq.(29) let us consider some special cases of low dimensions in more detail.

$D = 3$**:** Let us first of all consider a three-dimensional one-particle Hilbert space in which we choose the basis in such a way that the pure input state is identical with one of the basis vectors, say $|1\rangle$, i.e. $\rho_{in}(\mathbf{m}_0 = D\mathbf{A}_{11}) \equiv |1\rangle\langle 1|$. According to Eqs.(21) it is also convenient to introduce the pure, anti-symmetric two-particle states

$$|(ij)\rangle = \frac{1}{\sqrt{2}}(|ij\rangle - |ji\rangle) \tag{41}$$

with $i, j \in \{1, .., D\}$. Eq.(22) implies that for $D = 3$ the entangled output states resulting from the one-parameter family of universal quantum processes of Eq.(29) are convex sums of the two two-particle states

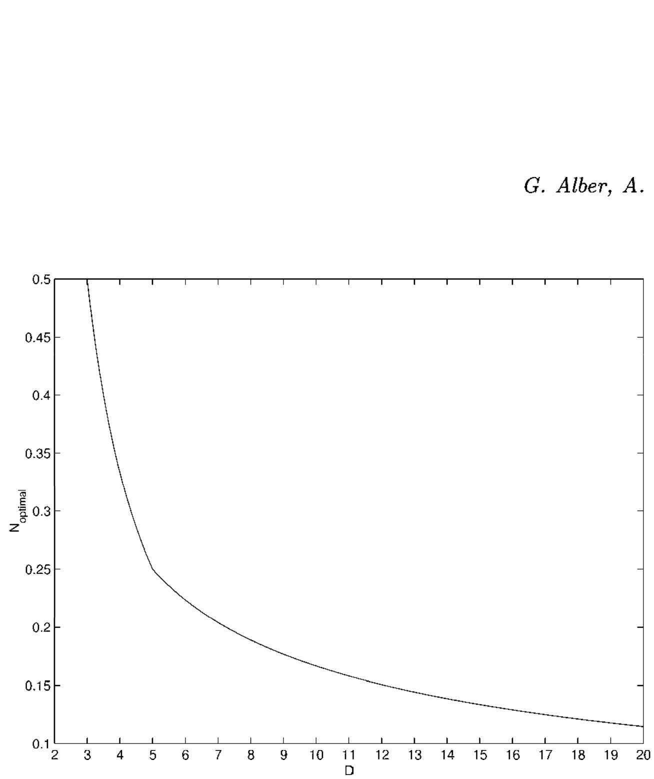

Fig. 5. Dimensional dependence of the entanglement measure $N(\rho)$ of Eq.(28) for the optimal universal entanglement processes discussed in Sec. 3.2.

$$\rho_2^{(ent)} = \frac{1}{2}\{|(12)\rangle\langle(12)| + |(13)\rangle\langle(13)|\},\ \rho_4^{(ent)} = |(23)\rangle\langle(23)|. \tag{42}$$

For $p_4 = 1$ the resulting output state $\rho_{out}^{(ent)}(\mathbf{m}_0 = D\mathbf{A}_{11}) = \rho_4^{(ent)}$ is pure so its von Neumann entropy vanishes. This state is the uniquely determined anti-symmetric, pure two-particle state which can be formed by the remaining two orthogonal basis states $|2\rangle$ and $|3\rangle$. Geometrically, this output state may be viewed as representing the uniquely determined plane which is orthogonal to the input state $|1\rangle$. This way this entangled output state preserves information about the input state. The index of correlation of this particular output state assumes the maximum possible value of $IC(p_4 = 1) = 2\ln 2$. The optimal universal entanglement process characterized by $p_2 = 1$ produces the mixed output state $\rho_{out}^{(ent)}(\mathbf{m} = D\mathbf{A}_{11}) = \rho_2^{(ent)}$. Its index of correlation also assumes the largest possible value of $IC(p_4 = 0) = 2\ln 2$. This optimal entanglement process maximizes the overlaps between the input state $|1\rangle$ and between the reduced one-particle states $R_1(\mathbf{m} = D\mathbf{A}_{11})$ and $R_2(\mathbf{m} = D\mathbf{A}_{11})$ of Eqs.(38). Thus, it preserves information about the initial input state in an optimal way. For universal entanglement process characterized $p_4 = 1/3$ ($p_1 = p_3 = 0$) the resulting output state is given by

$$\rho_{out}^{(ent)}(\mathbf{m} = D\mathbf{A}_{11}) = (2/3)\rho_2^{(ent)} + (1/3)\rho_4^{(ent)} \equiv$$
$$\frac{1}{3}\{|(12)\rangle\langle(12)| + |(13)\rangle\langle(13)| + |(23)\rangle\langle(23)|\}. \tag{43}$$

Its von Neumann entropy assumes the largest possible value of magnitude $S(p_4 = 1/3) = \ln 3$. This output state is a maximally disordered mixture of all possible anti-symmetric two-particle states which can be constructed from the underlying three-dimensional one-particle Hilbert spaces. In this particular universal entanglement process all information about the pure input state $|1\rangle$ is lost which is reflected by the fact that $\alpha^{(1)} = \alpha^{(2)} = 0$.

$D = 4$: For $D = 4$ the one-parameter family of output states of $\rho_{in}(\mathbf{m}_0 = D\mathbf{A}_{11}) \equiv |1\rangle\langle 1|$ is a convex sum of the two mixed states

$$\begin{aligned} \rho_2^{(ent)} &= \frac{1}{3}\{|(12)\rangle\langle(12)| + |(13)\rangle\langle(13)| + |(14)\rangle\langle(14)|\}, \\ \rho_4^{(ent)} &= \frac{1}{3}\{|(23)\rangle\langle(23)| + |(24)\rangle\langle(24)| + |(34)\rangle\langle(34)|\}. \end{aligned} \tag{44}$$

The optimal universal entanglement process with $p_2 = 1$ yields the mixed output state $\rho_{out}^{(ent)}(\mathbf{m} = D\mathbf{A}_{11}) = \rho_2^{(ent)}$ which implies maximal overlaps between the reduced one-particle states $R_1(\mathbf{m} = D\mathbf{A}_{11})$ and $R_2(\mathbf{m} = D\mathbf{A}_{11})$ of Eqs.(38) and the input state. Universal entanglement process with $p_4 = p_2 = 1/2$ produce a maximally disordered mixture of all possible anti-symmetric two particle states, i.e.

$$\begin{aligned} &\rho_{out}^{(ent)}(\mathbf{m} = D\mathbf{A}_{11}) = (1/2)\rho_2^{(ent)} + (1/2)\rho_4^{(ent)} \equiv \\ &\frac{1}{6}\{|(12)\rangle\langle(12)| + |(13)\rangle\langle(13)| + |(14)\rangle\langle(14)| + \\ &|(23)\rangle\langle(23)| + |(24)\rangle\langle(24)| + |(34)\rangle\langle(34)|\}. \end{aligned} \tag{45}$$

In this universal entanglement process all information about the input state is lost.

$D = 2$: Let us close with some final remarks concerning the special case of qubits for which some of the considerations of this chapter have to be modified. According to Eqs.(21) in this case $\rho_4^{(ent)} \equiv 0$ and thus disappears from Eq.(22). Consequently only one universal entanglement process is possible which does not yield any separable components. It is characterized by $p_1 = p_3 = p_4 = 0$ and by the pure, anti-symmetric output state

$$\rho_{out}^{(ent)}(\mathbf{m}_0 \equiv D\mathbf{A}_{11}) = |(12)\rangle\langle(12)|. \tag{46}$$

Thus, in this case the one-parameter family of universal entanglement processes of Eq.(29) collapses to a single process whose output state is independent of the input states.

4. Conclusions

It has been demonstrated that in Hilbert spaces of dimensions larger than two the linear character of quantum mechanics is compatible with the existence of optimal universal two-particle entanglement processes which preserve information about input states. This situation is completely different from the case of qubits where only one optimal universal two-particle entanglement process is possible in which all information about any input state is lost. The presented optimal universal entanglement processes are members of a one-parameter family of universal quantum processes which yield entangled output states without any separable components. Optimal universal entanglement processes involving two five dimensional quantum systems are exceptional in the sense that they coincide with this latter one-parameter family of universal entanglement processes. For all other dimensions the optimal universal entanglement process is one particular member of this one-parameter family of quantum processes. One of the characteristic features of this class of universal entanglement processes is that they always yield anti-symmetric output states which, with the single exception of qubit systems, preserve information about the input state.

The presented investigations indicate that convex sums of anti-symmetric quantum states resulting from the optimal universal entanglement processes discussed might also play an important role in universal entanglement processes which involve more than two quantum systems. Furthermore, entanglement processes which also preserve information about input states might have interesting applications in various branches of quantum information processing, such as quantum cryptography and quantum error correction. Thus, the presented results indicate that further exploration of quantum information processing beyond qubits may offer unexpected and useful surprises.

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