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ON IMPACT OF *A PRIORI* CLASSICAL KNOWLEDGE OF DISCRIMINATED STATES ON THE OPTIMAL UNAMBIGUOUS DISCRIMINATION

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Due to the fundamental limitations related to the Heisenberg uncertainty principle and the non-cloning theorem, it is impossible, even in principle, to determine the quantum state of a single system without *a priori* knowledge of it. To discriminate nonorthogonal quantum states in some optimal way, *a priori* knowledge of the discriminated states has to be relied upon. In this paper, we thoroughly investigate some impact of *a priori* classical knowledge of two quantum states on the optimal unambiguous discrimination. It is exemplified that *a priori* classical knowledge of the discriminated states, incomplete or complete, can be utilized to improve the optimal success probabilities, whereas the lack of *a prior* classical knowledge can not be compensated even by more resources.

Keywords: quantum state discrimination, a priori knowledge, unambiguous discrimination
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1 Introduction

In quantum information and quantum computation, the carrier of information is a quantum system in whose state information is encoded[1]. Since the state is not an observable in quantum mechanics[2], its discrimination is a fundamental problem in quantum information processing[3]. Hence two basic measurement strategies have been proposed, i.e., minimum-error discrimination[4] and optimum unambiguous discrimination[5, 6, 7, 8]. Researchers have explored ways to develop programmable quantum devices[9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20], especially quantum state discriminators[9, 19, 20], to accomplish quantum information processing tasks. However, due to the fundamental limitations related to the

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Heisenberg uncertainty principle[21] and the non-cloning theorem[22], it is not possible, even in principle, to determine the quantum state of a single system without *a priori* knowledge[23]. Optimal discrimination between nonorthogonal quantum states depends on the full utilization of *a priori* knowledge.

In their pioneering work[9], J. Bergou and M. Hillery constructed a device that can unambiguously discriminate between two unknown quantum states. J. Bergou et.al[20] further show how to construct devices that can optimally discriminate between one known and one unknown state using both minimum-error and optimum unambiguous strategies. It has also been recognized by J. Bergou et.al[9, 20] that the failure probability can be somewhat reduced (and the success probability can be somewhat improved) when one of the reference states is known. However, the impact of complete or incomplete *a priori* classical knowledge of discriminated states on the optimum unambiguous discrimination has not been investigated thoroughly.

To analyze such an impact, four cases will be taken into consideration: (1) we have no classical knowledge of either state; (2) we have classical knowledge of only one of two states; (3) we have classical knowledge of one of the states and the absolute value of the inner product of two states without the classical knowledge of the other; (4) we have classical knowledge of both states. For the sake of comparative analysis, the optimum success probabilities are viewed as the function of both preparation probabilities and the absolute value of the inner product of the two discriminated states. From this point of view, Refs. [9] and [8] provide us with excellent analysis bases for the first and the fourth case, respectively. In terms of the second case, however, current results in Ref. [20] are not suitable for comparative analysis. In fact, the optimal performance Eq. (2.19) of Ref. [20] is given in the average sense, thus the optimum failure/success probabilities only depend on the preparation probabilities of the two discriminated states rather than on both the preparation probabilities and the abstract value of the inner product of them. To the authors' knowledge, there has not been any serious investigation in the optimal unambiguous discrimination for the third case. The effect of *a priori* classical knowledge of the absolute value of the inner product of two states on optimal success probabilities has not been recognized and studied.

The rest of this paper is organized as follows. In Sect. II, we not only recall the results on optimal unambiguous discrimination for the first and fourth case, but also carefully study optimal unambiguous discrimination for the second and third case by viewing the optimum success probabilities as the function of both the preparation probabilities and the abstract value of the scalar product of two discriminated states. The comparative analysis and some discussions are further given in Sect. III. It is illustrated how *a priori* classical knowledge of the discriminated states, incomplete or complete, can be utilized to design optimal state discriminators in this section. The paper concludes with Sect. IV.

2 Optimal unambiguous discrimination problems for four cases

To analyze the impact of *a priori* classical knowledge of two discriminated states on both the design of optimum unambiguous discriminators and optimum success probabilities of discriminating two states, optimal unambiguous discrimination problems are discussed for the four cases in this section.

Case 1, we have no classical knowledge of either state;

Case 2, we only have classical knowledge of one of the two states;

Case 3, we have classical knowledge of one of the two states and the absolute value of the inner product of both states;

Case 4, we have classical knowledge of both states.

For simplicity, it is always assumed in this paper that one only has a single copy of an unknown state when one has no classical knowledge of this state.

2.1 the optimal unambiguous discrimination problem for Case 1

In this subsection, we just recall the result of Ref. [9] briefly.

Given two unknown quantum states $|\psi_1\rangle$ and $|\psi_2\rangle$, we can construct a device that will unambiguously discriminate between them. We shall consider the following problem which is perhaps the simplest version of a programmable state discriminator. Two classically unknown states $|\psi_1\rangle$ and $|\psi_2\rangle$ are provided as two inputs for two program registers, respectively. Then we are given another qubit that is guaranteed to be one of two unknown states $|\psi_1\rangle$ and $|\psi_2\rangle$ stored in the two program register. Our task is to determine, as best as we can, which one the given qubit is. We are allowed to fail, but not to make a mistake. What is the best procedure to accomplish this? An optimal unambiguous discriminator can be constructed according to Ref. [9] and the optimum success probability has been summarized as follows

$$P_0^{opt}(\beta,\eta_1) = \begin{cases} \frac{2}{3}[1-\sqrt{\eta_1(1-\eta_1)}](1-\beta^2) & \frac{1}{5} \le \eta_1 \le \frac{4}{5} \\ \frac{1}{2}(1-\eta_1)(1-\beta^2) & \eta_1 \le \frac{1}{5} \\ \frac{1}{2}\eta_1(1-\beta^2) & \eta_1 \ge \frac{4}{5} \end{cases} \tag{1}$$

where η_1 is the preparation probability of $|\psi_1\rangle$, $|\langle\psi_1|\psi_2\rangle| = \beta$ and the subscript 0 of P_0^{opt} means that we have no *a priori* classical knowledge of $|\psi_1\rangle$ and $|\psi_2\rangle$. In terms of this case, the measurement is performed on 3-qubit and the detection operators can be constructed as per Ref. [9].

2.2 the optimal unambiguous discrimination problem for Case 2

In this subsection, we re-discuss the optimal unambiguous discrimination problem for the second case and obtain some new results, which are different from those in Ref. [20].

Given one known quantum state $|\psi_1\rangle$ and one unknown quantum state $|\psi_2\rangle$, we can construct a device that unambiguously discriminate between them. We shall consider the following problem which may be a simple version of a programmable state discriminator. The unknown state $|\psi_2\rangle$ is provided as an input for the program register. Then we are given another qubit that is guaranteed to be in the known state $|\psi_1\rangle$ or the unknown state $|\psi_2\rangle$ stored in the program register. Our task is to determine, as best as we can, which one the given qubit is. As in case 1, we are allowed to fail, but not to make a mistake. What is the best procedure to accomplish this?

In line with Ref. [9], one can construct such a device by viewing this problem as a task in measurement optimization. The measurement is allowed to return an inconclusive result but never an erroneous one. Thus, it will be described by a POVM (positive-operator-valued measure) that will return outcome 1 (the unknown state stored in the data register matches the known state $|\psi_1\rangle$), 2 (the unknown state stored in the data register matches $|\psi_2\rangle$ stored in the program register), or 0 (we do not learn anything about the unknown state stored in the data register). Our task is then reduced to the following measurement optimization problem.

One has two input states

$$|\Psi_1^{in}\rangle = |\psi_2\rangle_A|\psi_1\rangle_B;\ |\Psi_2^{in}\rangle = |\psi_2\rangle_A|\psi_2\rangle_B \tag{2}$$

where the subscript A refers to the program register (A contains $|\psi_2\rangle$), and the subscript B refers to the data register. Our goal is to unambiguously distinguish between these inputs.

Let the elements of our POVM be $\prod_1$, corresponding to unambiguously detecting $|\psi_1\rangle$, $\prod_2$, corresponding to unambiguously detecting $|\psi_2\rangle$, and $\prod_0$, corresponding to failure, respectively. The probabilities of successfully identifying the two possible input states are given by

$$\langle\Psi_1^{in}|\prod\nolimits_1|\Psi_1^{in}\rangle = p_1;\ \langle\Psi_2^{in}|\prod\nolimits_2|\Psi_2^{in}\rangle = p_2 \tag{3}$$

and the condition of no errors implies that

$$\prod\nolimits_1|\Psi_2^{in}\rangle = 0;\ \prod\nolimits_2|\Psi_1^{in}\rangle = 0 \tag{4}$$

In addition, because the alternatives represented by the POVM exhaust all possibilities, we have that

$$\prod\nolimits_1 + \prod\nolimits_2 + \prod\nolimits_0 = I \tag{5}$$

Since we know nothing about $|\psi_2\rangle$ but have the classical knowledge of $|\psi_1\rangle$, the right way of constructing POVM operators is to take advantage of the symmetrical properties of the state as well as the classical knowledge of $|\psi_1\rangle$. Denoting $|\psi_1^{\perp}\rangle$ as the unit vector orthogonal to $|\psi_1\rangle$, we define the antisymmetric state

$$|\psi_{AB}^{-}\rangle = \frac{1}{\sqrt{2}}(|\psi_1\rangle_A|\psi_1^{\perp}\rangle_B - |\psi_1^{\perp}\rangle_A|\psi_1\rangle_B) \tag{6}$$

and introduce the projectors to the antisymmetric subspaces of the corresponding qubit as

$$P_{AB}^{as} = |\psi_{AB}^{-}\rangle\langle\psi_{AB}^{-}| \tag{7}$$

Furthermore, in terms of $|\psi_1\rangle$, $|\psi_1^{\perp}\rangle$ and the absolute value of inner product $\beta = |\langle\psi_1|\psi_2\rangle|$, we can obtain $|\psi_2\rangle = e^{i\delta}\beta|\psi_1\rangle + \sqrt{1-\beta^2}|\psi_1^{\perp}\rangle$ where δ and β are unknown real in this case and $0 \le \beta \le 1$.

Although β is unknown in this case, it can be clearly illustrated from the geometric point of view. Therefore, we still consider the success probabilities of unambiguously discriminating two states as a function of both the preparation probabilities and the absolute value of inner product β of of two discriminated states.

By making full use of the knowledge of $|\psi_1\rangle$ and $|\psi_1^{\perp}\rangle$, we construct the measurement operators $\prod_1$ and $\prod_2$ to satisfy the no-error condition (4) as follows:

$$\prod\nolimits_1 = c_1 P_{AB}^{as} \tag{8}$$

and

$$\prod\nolimits_2 = c_2|\psi_1\rangle_A|\psi_1^{\perp}\rangle_B{}_B\langle\psi_1^{\perp}|_A\langle\psi_1| + c_3|\psi_1^{\perp}\rangle_A|\psi_1^{\perp}\rangle_B{}_B\langle\psi_1^{\perp}|_A\langle\psi_1^{\perp}| \tag{9}$$

where c_1, c_2 and c_3 are as yet undetermined nonnegative real numbers. Using the Eqs. (8) and (9), we have

$$p_1 = \langle\Psi_1^{in}|\prod\nolimits_1|\Psi_1^{in}\rangle = \frac{1}{2}c_1(1-\beta^2) \tag{10}$$

$$p_2 = \langle \Psi_2^{in} | \prod\nolimits_2 | \Psi_2^{in} \rangle = c_2\beta^2(1-\beta^2) + c_3(1-\beta^2)^2 \tag{11}$$

By assuming that the preparation probabilities of $|\psi_1\rangle$ and $|\psi_2\rangle$ are η_1 and η_2, respectively, we can define the average probability P of successfully discriminating two states as

$$P = [\frac{1}{2}c_1\eta_1 + c_2\beta^2\eta_2 + c_3(1-\beta^2)\eta_2](1-\beta^2) \tag{12}$$

where $\beta = |\langle\psi_1|\psi_2\rangle|$, and our task is to maximize the performance (12) subject to the constraint that $\prod_0 = I - \prod_1 - \prod_2$ is a positive operator. It can be demonstrated that one cannot maximize (12) everywhere simultaneously without the classical knowledge of β. Still, we can give some further analysis.

To assure that $\prod_0$, $\prod_1$ and $\prod_2$ are positive operator, we have the following inequality constraints:

$$0 \le c_1 \le 1; 0 \le c_2 \le 1; 0 \le c_3 \le 1 \tag{13}$$

and

$$1 - c_1 - c_2 + \frac{1}{2}c_1c_2 \ge 0 \tag{14}$$

In any case, one can always choose $c_3^{opt} = 1$ to make $c_3(1-\beta^2)\eta_2$ as large as possible; thus, the aforementioned problem can be reduced to choosing c_1 and c_2 to "maximize" $\frac{1}{2}c_1\eta_1 + c_2\beta^2\eta_2$ subject to the constraints on c_1 and c_2 from some different viewpoints. Here, we just adopt what can be called a hybrid strategy. Several alternative strategies will also be discussed in the remarks of this subsection. The advantage of this hybrid strategy will be further illustrated in the next section.

Subsequently, we will explain our strategy as follows.

First, we rewrite the performance (12) as

$$P = [\frac{1}{2}c_1\eta_1 + c_2\eta_2 + (c_3 - c_2)(1-\beta^2)\eta_2](1-\beta^2) \tag{15}$$

Second, we choose $c_3^{opt} = 1$.

Third, we design c_1^{opt} and c_2^{opt} to maximize

$$Q = \frac{1}{2}c_1\eta_1 + c_2\eta_2 \tag{16}$$

subject to the following constraints: $0 \le c_1 \le 1$, $0 \le c_2 \le 1$ and (14).

After some calculations, we obtain the optimum values of c_1 and c_2 as follows:

If $\frac{1}{2} \le \eta_1 \le \frac{4}{5}$ (i.e., $\frac{1}{5} \le \eta_2 \le \frac{1}{2}$), then $c_1^{opt} = 2(1 - \sqrt{\frac{\eta_2}{\eta_1}})$ and $c_2^{opt} = 2 - \sqrt{\frac{\eta_1}{\eta_2}}$. Inserting these optimal values into (16) gives

$$Q_{POVM} = (2 - \eta_1 - 2\sqrt{\eta_1\eta_2}) \tag{17}$$

If $\eta_1 \ge \frac{4}{5}$ (i.e., $\eta_2 \le \frac{1}{5}$), then $c_1^{opt} = 1$ and $c_2^{opt} = 0$. These values yield the following equation

$$Q_1 = \frac{1}{2}\eta_1 \tag{18}$$

If $\eta_2 \geq \frac{1}{2}$ (i.e., $\eta_1 \leq \frac{1}{2}$), then $c_1^{opt} = 0$ and $c_2^{opt} = 1$. Substituting these optimal values into (16) gives

$$Q_2 = \eta_2 \tag{19}$$

Now the performance (16) can be summarized as

$$Q_1^{opt}(\eta_1) = \begin{cases} [2 - \eta_1 - 2\sqrt{\eta_1(1-\eta_1)}] & \frac{1}{2} \leq \eta_1 \leq \frac{4}{5} \\ (1-\eta_1) & \eta_1 \leq \frac{1}{2} \\ \frac{1}{2}\eta_1 & \eta_1 \geq \frac{4}{5} \end{cases} \tag{20}$$

Finally, by substituting c_1^{opt}, c_2^{opt} and c_3^{opt} into (15), we obtain the hybrid optimum success probability $P_1^{Hopt}(\beta, \eta_1) = [Q_1^{opt}(\eta_1) + (1 - c_2^{opt})(1-\beta^2)\eta_2](1-\beta^2)$. which can be futher summarized as:

$$P_1^{Hopt}(\beta, \eta_1) = \begin{cases} [1 + \beta^2(1-\eta_1) - (1+\beta^2)\sqrt{\eta_1(1-\eta_1)}](1-\beta^2) & \frac{1}{2} \leq \eta_1 \leq \frac{4}{5} \\ (1-\eta_1)(1-\beta^2) & \eta_1 \leq \frac{1}{2} \\ (1 - \frac{1}{2}\eta_1 - \beta^2(1-\eta_1))(1-\beta^2) & \eta_1 \geq \frac{4}{5} \end{cases} \tag{21}$$

where the subscript 1 of P_1^{Hopt} means that we just have *a priori* classical knowledge of $|\psi_1\rangle$, one of two discriminated states, and the superscript $Hopt$ implies that the hybrid optimum success probability is only obtained by the aforementioned hybrid strategy.

Remark:

1. When the classical knowledge of β is not available, an alternative strategy is to reconsider this problem in the sense of the worst case as suggested in Ref. [20]. In general, we can use the worst-case strategy to optimize the success probability subject to the constraints (13) and (14). The worst-case strategy can be clearly explained from the geometric point of view. Under this circumstance, the constrained optimum performance is no longer a function of both β and η_1.

2. The worst-case strategy can also be used to "maximize" $\frac{1}{2}c_1\eta_1 + c_2\beta^2\eta_2$ subject to the constraints $0 \leq c_1 \leq 1$, $0 \leq c_2 \leq 1$ and (14). After some calculations, we have $c_1^W = 1$ and $c_2^W = 0$. By inserting these values and $c_3^{opt} = 1$ into (12), we have

$$P_1^W(\beta, \eta_1) = (1 - \frac{1}{2}\eta_1 - \beta^2(1-\eta_1))(1-\beta^2) \tag{22}$$

where the subscript 1 of P_1^W means that we just have *a priori* classical knowledge of $|\psi_1\rangle$, one of two discriminated states, and the superscript W means that we design c_1 and c_2 to maximize $\frac{1}{2}c_1\eta_1 + c_2\beta^2\eta_2$ in the sense of the worst case. This hybrid worst-case strategy will be further illustrated in the subsection 2.3. However, the disadvantage of this strategy is that $P_1^W(\beta, \eta_1)$ is worse than $P_0^{opt}(\beta, \eta_1)$ given in (1) somewhere.

3. Another alternative strategy is to maximize the performance $\frac{1}{2}c_1\eta_1 + c_2\beta^2\eta_2$ subject to some additional constraints. For example, when we choose $c_1 = c_2$ as an additional constraint, we have $c_1^* = c_2^* = 2 - \sqrt{2}$. Substituting c_1^*, c_1^* and $c_3^{opt} = 1$ into (12), we have

$$P_1^*(\beta, \eta_1) = [1 - \frac{\sqrt{2}}{2}\eta_1 + (1-\sqrt{2})\beta^2(1-\eta_1)](1-\beta^2) \tag{23}$$

where the subscript 1 of P_1^* means that we just have *a priori* classical knowledge of $|\psi_1\rangle$, one of two discriminated states, and the superscript $*$ means that we obtain the special performance

with the additional constrain $c_1 = c_2$. This special strategy will also be further explained in the next subsection. Unfortunately, $P_1^*(\beta, \eta_1)$ fails to be better than $P_0^{opt}(\beta, \eta_1)$ given in (1) everywhere.

4. The problem considered in this subsection is quite different from the unambiguous discrimination between a known pure state and a mixed state. In the average case, the unambiguous discrimination between a known and an unknown pure state can be converted into the one between a known pure state and a mixed state as in Ref. [20]. However, it is not necessary to make such a conversion in this situation.

5. In terms of this case, the generalized measurement is performed on the two qubits.

2.3 the optimal unambiguous discrimination problem for Case 3

To our knowledge, the optimal unambiguous discrimination problem had not been investigated for the third case before. In this subsection, we present our results for this case.

If one state $|\psi_1\rangle$ is known with extra classical knowledge that $|\langle\psi_1|\psi_2\rangle| = \beta$, whereas the quantum state $|\psi_2\rangle$ itself is unknown, we can construct a device that will unambiguously discriminate between them. In terms of $|\psi_1\rangle$, $|\psi_1^\perp\rangle$ and the absolute value of inner product $\beta = |\langle\psi_1|\psi_2\rangle|$, we can obtain $|\psi_2\rangle = e^{i\delta}\beta|\psi_1\rangle + \sqrt{1-\beta^2}|\psi_1^\perp\rangle$ where δ and β are unknown real in this case. It is obvious that $0 \le \beta \le 1$. With *a priori* classical knowledge of $|\langle\psi_1|\psi_2\rangle| = \beta$ in hand, one still chooses the operators $\prod_1$ and $\prod_2$ to satisfy the no-error condition (4) as follows:

$$\prod\nolimits_1 = c_1 P_{AB}^{as} \tag{24}$$

and

$$\prod\nolimits_2 = c_2|\psi_1\rangle_A|\psi_1^\perp\rangle_{B\,B}\langle\psi_1^\perp|_A\langle\psi_1| + c_3|\psi_1^\perp\rangle_A|\psi_1^\perp\rangle_{B\,B}\langle\psi_1^\perp|_A\langle\psi_1^\perp| \tag{25}$$

with $0 \le c_1, c_2, c_3 \le 1$. Using the Eqs. (24)-(25), we get

$$p_1 = \langle\Psi_1^{in}|\prod\nolimits_1|\Psi_1^{in}\rangle = \frac{1}{2}c_1(1-\beta^2) \tag{26}$$

and

$$p_2 = \langle\Psi_2^{in}|\prod\nolimits_2|\Psi_2^{in}\rangle = c_3(1-\beta^2)^2 + c_2\beta^2(1-\beta^2) \tag{27}$$

By denoting the preparation probabilities of $|\psi_1\rangle$ and $|\psi_2\rangle$ as η_1 and η_2, respectively, we can define the average probability P of successfully discriminating two states as

$$P = [\frac{1}{2}c_1\eta_1 + c_2\beta^2\eta_2 + c_3(1-\beta^2)\eta_2](1-\beta^2) \tag{28}$$

Now our task is reduced to maximizing the performance (28) subject to the constraint that $\prod_0 = I - \prod_1 - \prod_2$ is a positive operator. In other words, our task is to design c_1^{opt}, c_2^{opt} and c_3^{opt} to maximize the performance (28) subject to the constraints (13) and (14).

With *a priori* classical knowledge of $|\langle\psi_1|\psi_2\rangle| = \beta$ in hand, we can get the optimum values c_1^{opt}, c_2^{opt} and c_3^{opt} as follows:

If $\frac{\beta^2}{1+\beta^2} \le \eta_1 \le \frac{4\beta^2}{1+4\beta^2}$, then $c_1^{opt} = 2(1-\beta\sqrt{\frac{\eta_2}{\eta_1}})$, $c_2^{opt} = 2 - \frac{1}{\beta}\sqrt{\frac{\eta_1}{\eta_2}}$ and $c_3^{opt} = 1$.

If $\eta_1 \le \frac{\beta^2}{1+\beta^2}$, then $c_1^{opt} = 0$ and $c_2^{opt} = c_3^{opt} = 1$.

If $\eta_1 \ge \frac{4\beta^2}{1+4\beta^2}$, then $c_1^{opt} = c_3^{opt} = 1$ and $c_2^{opt} = 0$.

Taking Eq. (28) into consideration, we obtain the corresponding optimum success probabilities:

$$P_{1+}^{opt}(\beta,\eta_1) = \begin{cases} [1+\beta^2(1-\eta_1)-2\beta\sqrt{\eta_1(1-\eta_1)}](1-\beta^2) & \frac{\beta^2}{1+\beta^2} \le \eta_1 \le \frac{4\beta^2}{1+4\beta^2} \\ (1-\eta_1)(1-\beta^2) & \eta_1 \le \frac{\beta^2}{1+\beta^2} \\ (1-\frac{1}{2}\eta_1-\beta^2(1-\eta_1))(1-\beta^2) & \eta_1 \ge \frac{4\beta^2}{1+4\beta^2} \end{cases} \tag{29}$$

where the subscript 1+ of P_{1+}^{opt} means that we have *a priori* classical knowledge of one of the two discriminated states and the absolute value of the inner product of the two states.

Remark:

1. Because a priori knowledge of β is available in this case, the optimal performance (29) is better than the hybrid optimum probability (21) in **Case** 2. When $\beta = 1$ (or tends to one, if we do not want to work with the trivial case of identical states), the POVM is the same as in Case 2, for every η_1. It might be interesting to highlight the fact that the POVMs are also exactly the same in cases 2 and 3 when η_1 is near either 1 or 0. This will be further discussed in section 3.

2. As long as $\eta_1 \ge \frac{4\beta^2}{1+4\beta^2}$, we should choose $c_1^{opt} = c_3^{opt} = 1$ and $c_2^{opt} = 0$ from the above analysis. Substituting these values into (12) yields $P_1^W(\beta,\eta_1)$ given in (22). It is easy to show that the optimal performance $P_{1+}^{opt}(\beta,\eta_1)$ is better than $P_1^W(\beta,\eta_1)$.

3. When $\beta = \frac{\sqrt{2}}{2}$ and $\eta_1 = \frac{1}{3}$, we have $c_1^{opt} = c_2^{opt} = 2-\sqrt{2}$ and $c_3^{opt} = 1$. Inserting these values into (12) yields $P_1^*(\beta,\eta_1)$ given in (23). It can also be demonstrated that the optimal performance $P_{1+}^{opt}(\beta,\eta_1)$ is better than $P_1^*(\beta,\eta_1)$.

4. It should be pointed out that both the problem and the result presented here are quite different from the unambiguous overlap filtering described in [25]. In the unambiguous overlap filtering, the aim is to decide unambiguously whether or not $|\langle\psi_1|\psi_2\rangle| = \beta$ for some arbitrary fixed β. However, our aim here is to unambiguously discriminate between $|\psi_1\rangle$ and $|\psi_2\rangle$ with the classical knowledge of $|\psi_1\rangle$ and β in hand.

5. In terms of this case, the generalized measurement is performed on the two qubits.

2.4 the optimal unambiguous discrimination problem for **Case** 4

In this subsection, we recall the result of Ref. [3] or Ref. [8] for the fourth case.

If we have complete *a priori* information of both $|\psi_1\rangle$ and $|\psi_2\rangle$, then we can write the optimum success probability for this case as per Ref. [3, 8]

$$P_2^{opt}(\beta,\eta_1) = \begin{cases} 1-2\sqrt{\eta_1(1-\eta_1)}\beta & \frac{\beta^2}{1+\beta^2} \le \eta_1 \le \frac{1}{1+\beta^2} \\ (1-\eta_1)(1-\beta^2) & \eta_1 \le \frac{\beta^2}{1+\beta^2} \\ \eta_1(1-\beta^2) & \eta_1 \ge \frac{1}{1+\beta^2} \end{cases} \tag{30}$$

where $|\langle\psi_1|\psi_2\rangle| = \beta$, the subscript 2 of P_2^{opt} means that we have the classical knowledge of both discriminated states. In terms of this case, the measurement is performed on the detected qubit. One can select the detection operators as

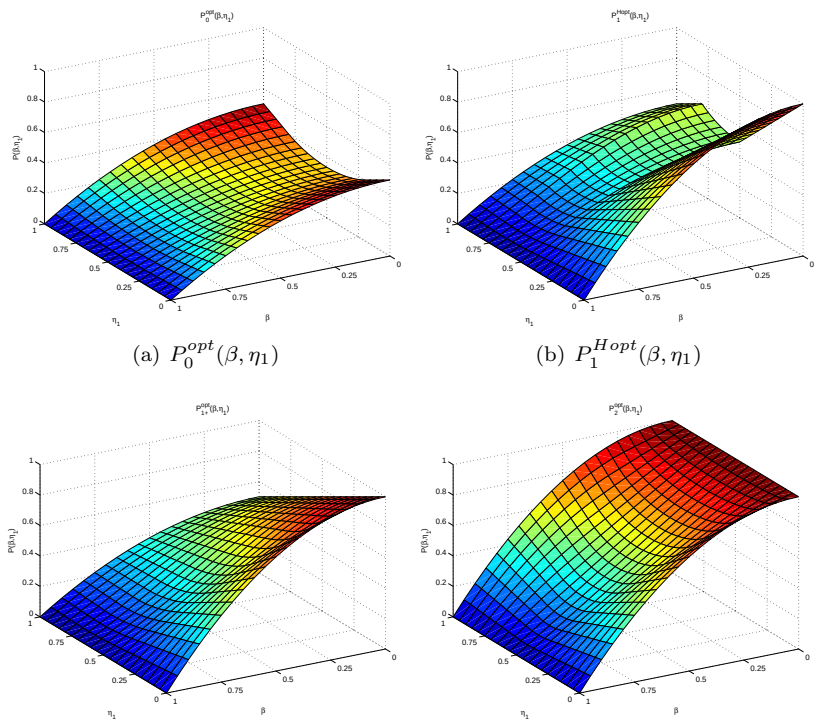
$$\prod\nolimits_1 = c_1|\psi_2^{\perp}\rangle\langle\psi_2^{\perp}|;\ \prod\nolimits_2 = c_2|\psi_1^{\perp}\rangle\langle\psi_1^{\perp}| \tag{31}$$

where the optimum values of c_1 and c_2 can be obtained easily.

3 Comparative analysis and some discussions

In this section, we make the comparative analysis based on Eqs.(1), (21), (29) and (30), and discuss the effect of a priori classical knowledge of the discriminated states on optimum success probabilities.

Let us consider the optimum success probabilities for the four cases: The functions $P_0^{opt}(\beta,\eta_1)$, $P_1^{Hopt}(\beta,\eta_1)$, $P_{1+}^{opt}(\beta,\eta_1)$ and $P_2^{opt}(\beta,\eta_1)$ are shown in Fig.1(a)∼Fig.1(d), respectively. Some remarks and illustrations are given as follows.



(a) $P_0^{opt}(\beta,\eta_1)$ (b) $P_1^{Hopt}(\beta,\eta_1)$

(c) $P_{1+}^{opt}(\beta,\eta_1)$ (d) $P_2^{opt}(\beta,\eta_1)$

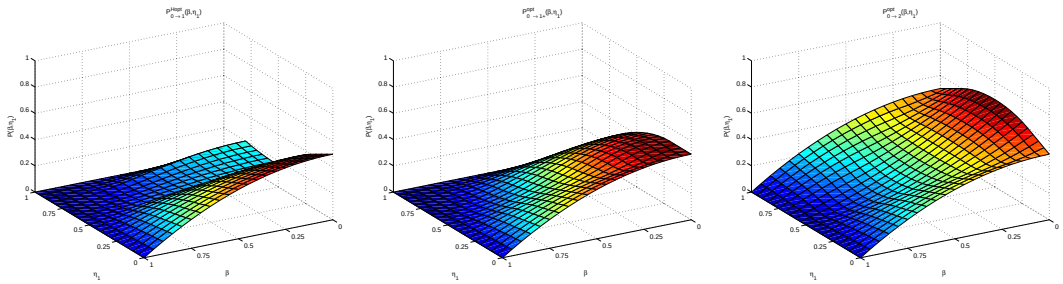
Fig. 1. Optimum success probabilities: $P_0^{opt}(\beta,\eta_1)$, $P_1^{Hopt}(\beta,\eta_1)$, $P_{1+}^{opt}(\beta,\eta_1)$ and $P_2^{opt}(\beta,\eta_1)$

1. To compare Fig.1(a) with Fig.1(b)∼Fig.1(d), we further plot three performance-improving functions $P_{0\to1}^{Hopt}(\beta,\eta_1)=P_1^{Hopt}(\beta,\eta_1)-P_0^{opt}(\beta,\eta_1)$, $P_{0\to1+}^{opt}(\beta,\eta_1)=P_{1+}^{opt}(\beta,\eta_1)-P_0^{opt}(\beta,\eta_1)$ and $P_{0\to2}^{opt}(\beta,\eta_1)=P_2^{opt}(\beta,\eta_1)-P_0^{opt}(\beta,\eta_1)$ in Fig. 2(a)∼Fig. 2(c), respectively.

(i) By comparing $P_0^{opt}(\beta,\eta_1)$ with $P_1^{Hopt}(\beta,\eta_1)$(see Fig. 2(a)), we demonstrate that *a priori* classical knowledge of one of both states can be cleverly used to improve the optimum success probabilities. When both β and the preparation probability η_1 are small, the classical knowledge of one of two states can be used to significantly improve the optimum success probabilities. This is in contrast to the fact that both $P_1^M(\beta,\eta_1)$ and $P_1^*(\beta,\eta_1)$ are worse than $P_0^{opt}(\beta,\eta_1)$ somewhere.

(ii) More interestingly, it is shown that *a priori* classical knowledge of one of two states and the absolute value of inner product of two states can always improve the optimum performance to a varying extent. especially when both β and η_1 are small (See Fig.2(b)).

(iii) Comparing $P_2^{opt}(\beta,\eta_1)$ with $P_0^{opt}(\beta,\eta_1)$, we can see from Fig.2(c) that *a priori* classical knowledge of both states will certainly improve the optimum performance except for the trivial case of $\beta = 1$. With *a priori* classical knowledge of both states in hand, one can always significantly improve the optimum success possibilities when β is small.



(a) $P_{0\to1}^{Hopt}(\beta,\eta_1)$ (b) $P_{0\to1+}^{opt}(\beta,\eta_1)$ (c) $P_{0\to2}^{opt}(\beta,\eta_1)$

Fig. 2. Performance-improving functions $P_{0\to1}^{Hopt}(\beta,\eta_1)$, $P_{0\to1+}^{opt}(\beta,\eta_1)$ and $P_{0\to2}^{opt}(\beta,\eta_1)$

2. To evaluate the effect of *a priori* knowledge of β on the optimum success probabilities, we plot the performance-improving function $P_{1\to1+}^{Hopt} = P_{1+}^{opt}(\beta,\eta_1) - P_1^{Hopt}(\beta,\eta_1)$ in Fig. 3(a). To compare Fig. 1(d) with Fig. 1(b) and Fig. 1(c) intuitively, the performance-improving functions $P_{1\to2}^{Hopt}(\beta,\eta_1) = P_2^{opt}(\beta,\eta_1) - P_1^{Hopt}(\beta,\eta_1)$ and $P_{1+\to2}^{opt}(\beta,\eta_1) = P_2^{opt}(\beta,\eta_1) - P_{1+}^{opt}(\beta,\eta_1)$ are also shown in Fig. 3(b) and Fig. 3(c), respectively.

(i) By comparing $P_{1+}^{opt}(\beta,\eta_1)$ with $P_1^{Hopt}(\beta,\eta_1)$(see Fig.3(a)), one can reveal that *a priori* knowledge of β can be applied to improve the optimum success probabilities in principle, especially when η_1 is near 0.5 and β is smaller than 0.5. In addition, if η_1 is near 0 or 1, or if β is larger than 0.8; then $P_1^{Hopt}(\beta,\eta_1)$ is almost as good as $P_{1+}^{opt}(\beta,\eta_1)$. This not only clearly illustrates the impact of the classical knowledge of β on the optimum success probabilities, but also further explains that the hybrid strategy in this paper is a good choice.

(ii) By comparing $P_1^{Hopt}(\beta,\eta_1)$ with $P_2^{opt}(\beta,\eta_1)$(see Fig. 3(b)), one can observe that $P_2^{opt}(\beta,\eta_1)$ is not only always larger than $P_1^{Hopt}(\beta,\eta_1)$, but $P_2^{opt}(\beta,\eta_1)$ is also significantly better than $P_1^{Hopt}(\beta,\eta_1)$ especially when β is small and the preparation probability η_1 of $|\psi_1\rangle$ is large.

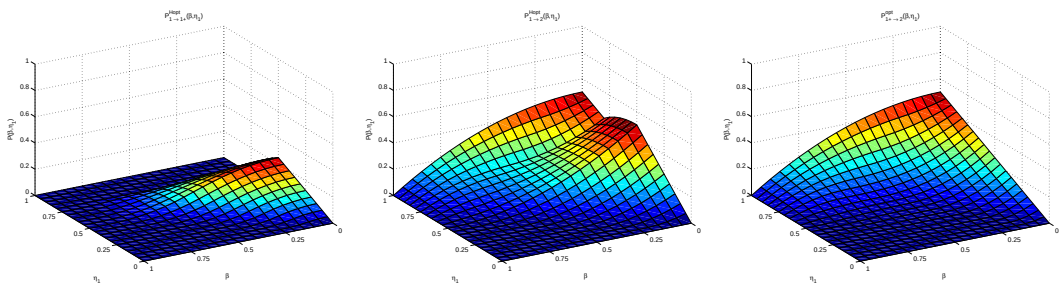
(iii) By comparing $P_{1+}^{opt}(\beta,\eta_1)$ with $P_2^{opt}(\beta,\eta_1)$(see Fig.3(c)), we can find that the optimum success possibilities can be improved significantly if β is small and η_1 is large. It is also interesting to find that $P_{1+}^{opt}(\beta,\eta_1)$=$P_2^{opt}(\beta,\eta_1)$ if $\eta_1 \le \frac{\beta^2}{1+\beta^2}$. That is to say, one with *a priori* classical knowledge of one of two states $|\psi_1\rangle$ and the absolute value of inner product of two states can obtain as good performance as another one with *a priori* classical knowledge of both states if the preparation probability η_1 of $|\psi_1\rangle$ is not larger than $\frac{\beta^2}{1+\beta^2}$.

3. To achieve a more in-depth view on the effect of *a priori* classical knowledge of the discriminated states on the optimum success probabilities, we take the following steps:

By setting $\eta_1 = \eta_2 = \frac{1}{2}$, the four functions of β from Eqs.(1), (21), (29) and (30) are obtained:

(1) If we have no classical knowledge about $|\psi_1\rangle$ and $|\psi_2\rangle$, then we have $P_0^{opt}(\beta) = \frac{1}{3}(1-\beta^2)$.

(2) If we have the classical knowledge of $|\psi_1\rangle$, then $P_1^{Hopt}(\beta) = \frac{1}{2}(1-\beta^2)$.



(a) $P^{Hopt}_{1\to 1+}(\beta,\eta_1)$ (b) $P^{Hopt}_{1\to 2}(\beta,\eta_1)$ (c) $P^{opt}_{1+\to 2}(\beta,\eta_1)$

Fig. 3. The performance-improving functions: $P^{Hopt}_{1\to 1+}(\beta,\eta_1)$, $P^{Hopt}_{1\to 2}(\beta,\eta_1)$ and $P^{opt}_{1+\to 2}(\beta,\eta_1)$.

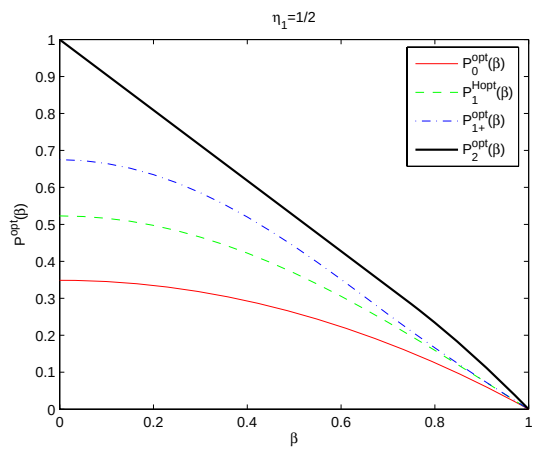


Fig. 4. By fixing the $\eta_1 = \frac{1}{2}$ in $P^{opt}_0(\beta,\eta_1)$, $P^{Hopt}_1(\beta,\eta_1)$, $P^{opt}_{1+}(\beta,\eta_1)$ and $P^{opt}_2(\beta,\eta_1)$, we get the functions $P^{opt}_0(\beta)$, $P^{Hopt}_1(\beta)$, $P^{opt}_{1+}(\beta)$ and $P^{opt}_2(\beta)$ and their images.

(3) If we have the classical knowledge of $|\psi_1\rangle$ and β, then we get

$$P^{opt}_{1+}(\beta) = \begin{cases} (\frac{3}{4} - \frac{1}{2}\beta^2)(1-\beta^2) & \beta \le \frac{1}{2} \\ (1 + \frac{1}{2}\beta^2 - \beta)(1-\beta^2) & \beta \ge \frac{1}{2} \end{cases} \tag{32}$$

(4) If we have the classical knowledge of both $|\psi_1\rangle$ and $|\psi_2\rangle$, then $P^{opt}_2(\beta) = 1-\beta$.

Functions $P^{opt}_0(\beta)$, $P^{Hopt}_1(\beta)$, $P^{opt}_{1+}(\beta)$ and $P^{opt}_2(\beta)$ are plotted in Fig. 4. In this special instance, the optimum success probabilities strictly grow with the increased *a priori* knowledge of the discriminated states.

4. We can also obtain the following functions by substituting $\beta^2 = \frac{1}{2}$ into Eqs.(1),(21),(29) and (30):

$$P^{opt}_0(\eta_1) = \begin{cases} \frac{1}{3}[1 - \sqrt{\eta_1(1-\eta_1)}] & \frac{1}{5} \le \eta_1 \le \frac{4}{5} \\ \frac{1}{4}(1-\eta_1) & \eta_1 \le \frac{1}{5} \\ \frac{1}{4}\eta_1 & \eta_1 \ge \frac{4}{5} \end{cases} \tag{33}$$

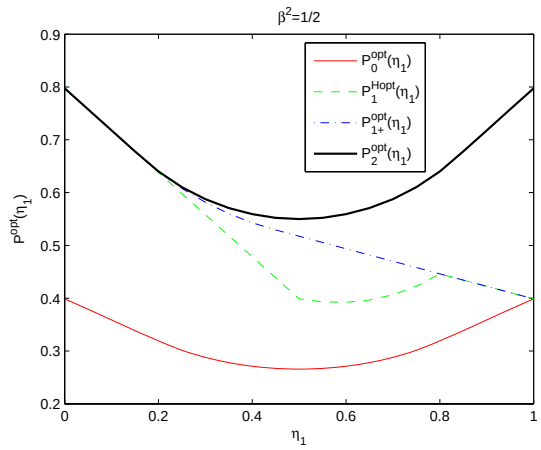


Fig. 5. By fixing the $\beta^2 = \frac{1}{2}$ in $P_0^{opt}(\beta, \eta_1)$, $P_1^{Hopt}(\beta, \eta_1)$, $P_{1+}^{opt}(\beta, \eta_1)$ and $P_2^{opt}(\beta, \eta_1)$, respectively, we get the functions $P_0^{opt}(\eta_1)$, $P_1^{Hopt}(\eta_1)$, $P_{1+}^{opt}(\eta_1)$ and $P_2^{opt}(\eta_1)$ and their images.

$$P_1^{Hopt}(\eta_1) = \begin{cases} \frac{1}{2} + \frac{1}{4}\eta_1 - \frac{3}{4}\sqrt{\eta_1(1-\eta_1)} & \frac{1}{2} \le \eta_1 \le \frac{4}{5} \\ \frac{1}{2}(1-\eta_1) & \eta_1 \le \frac{1}{2} \\ \frac{1}{4} & \eta_1 \ge \frac{4}{5} \end{cases} \tag{34}$$

$$P_{1+}^{opt}(\eta_1) = \begin{cases} \frac{3}{4} - \frac{1}{4}\eta_1 - \frac{1}{2}\sqrt{2\eta_1(1-\eta_1)} & \frac{1}{3} \le \eta_1 \le \frac{2}{3} \\ \frac{1}{2}(1-\eta_1) & \eta_1 \le \frac{1}{3} \\ \frac{1}{4} & \eta_1 \ge \frac{2}{3} \end{cases} \tag{35}$$

$$P_2^{opt}(\eta_1) = \begin{cases} 1 - \sqrt{2\eta_1(1-\eta_1)} & \frac{1}{3} \le \eta_1 \le \frac{2}{3} \\ \frac{1}{2}(1-\eta_1) & \eta_1 \le \frac{1}{3} \\ \frac{1}{2}\eta_1 & \eta_1 \ge \frac{2}{3} \end{cases} \tag{36}$$

as shown in Fig. 5.

To sum up, we can conclude that optimum success probabilities grow with increased *a priori* classical knowledge.

(1) When we have no classical knowledge of two discriminated states in the first case, we can make full use of symmetry to design an optimum unambiguous discriminator. Under this circumstance, we need two program registers.

(2) If we have incomplete a priori classical knowledge of two discriminated states in the second and third case, we can make full use of the classical knowledge of one of the two states or the absolute value of the inner product of both states to construct optimum unambiguous discriminators. In these two cases, we only need one program register.

(3) When we have full classical knowledge of two discriminated states in the fourth case, we can design an optimum unambiguous discriminator based on the classical knowledge of both states. In this case, we do not need any program register.

From the first case to the fourth case, we demonstrate that, with complete *a priori* classical knowledge about the discriminated states, one can obtain the best optimum success possibilities with the fewest resources (without any program register). This is in contrast with the observation that one can never gain better optimum success possibilities with incomplete or

no *a priori* classical knowledge in hand, even when more resources (one program register or two program registers) are utilized to construct optimum unambiguously discriminators.

4 Conclusion

In this paper, we have made a careful analysis on the impact of *a priori* classical knowledge of two discriminated states on the optimal unambiguous discrimination. Specially, we demonstrate the effect of *a priori* knowledge of β on optimal success probabilities where β is the absolute value of the inner product of two states. Moreover, we examine the role of *a priori* classical knowledge of two discriminated states in the design of optimum unambiguous discriminators, and investigate the effect of *a priori* knowledge on optimum success probabilities under four different circumstances. By using comparative analysis of optimum unambiguous discriminators for these four cases, it is exemplified that *a priori* classical knowledge of the discriminated states, incomplete or complete, can be utilized to improve the optimal success probabilities, whereas even more resources can not compensate the lack of *a priori* classical knowledge. This paper poses an interesting problem: what kind of quantum information is strictly equivalent to classical information for a given problem? In our opinion, it deserves further investigation in the future. This problem may recall us the famous observation that *no mathematical trick can compensate the lack of information.*

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