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PROTOCOL FOR UNAMBIGUOUS QUANTUM STATE DISCRIMINATION USING QUANTUM COHERENCE

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Roa *et al.* showed that quantum state discrimination between two nonorthogonal quantum states does not require quantum entanglement but quantum dissonance only. We find that quantum coherence can also be utilized for unambiguous quantum state discrimination. We present a protocol and quantify the required coherence for this task. We discuss the optimal unambiguous quantum state discrimination strategy in some cases. In particular, our work illustrates an avenue to find the optimal strategy for discriminating two nonorthogonal quantum states by measuring quantum coherence.

Keywords: Unambiguous quantum state discrimination(UQSD), Quantum coherence

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1 Introduction

A fundamental result in quantum mechanics is the impossibility to perfectly distinguish two or more nonorthogonal states. Quantum state discrimination (QSD) consists in devising strategies to discriminate nonorthogonal quantum states as accurately as possible. QSD has various useful applications in quantum information processing [1, 2, 3], and it branches out into two important streams: minimal-error deterministic quantum state discrimination (DQSD) [1] and unambiguous quantum state discrimination (UQSD) [4]. In DQSD, one always has an answer but with a probability of being wrong. On the other hand, in UQSD, one is guaranteed to never be wrong, but there are occasions when one does not have an answer. In UQSD, the task is to minimize the probability of no answer. Though several strategies exist to discriminate quantum states in the literature, optimal strategies of QSD are yet to be figured out in all the cases [5]. The study of minimization of error in state discrimination was pioneered by Helstrom [1] who provided a lower bound on the error probability for distinguishing two quantum states. It has been enriched further by presenting an upper bound of success probability for distinguishing arbitrary number of quantum states [6], and many studies have focused on achieving that bound [7, 8, 9, 10, 11, 12]. In addition, the protocol for unambiguous discrimination of linearly independent pure quantum states, assisted by an auxiliary system, is of fundamental interest [13]. While quantum entanglement [14] is regarded as a key resource in quantum information processing [15], other non-classical correlations such as quantum discord and quantum dissonance [16, 17, 18]) are also very useful. The assisted unambiguous discrimination for two nonorthogonal states that requires only quantum dissonance (zero entanglement and nonzero discord) was introduced by Roa *et al.* [19], and its generalization and various applications have been studied thereafter [20, 21]. An optical implementation of unambiguous discrimination of the two finite ensembles of coherent states was also proposed by Sedlák [22]. In this paper, we find a UQSD protocol that requires only quantum coherence as a resource.

Although those have intrinsically the same origin, viz. the superposition principle, more attention has been paid on the effects of entanglement and other quantum correlations than on the impact of quantum coherence [23, 24] on quantum advantages in devices and protocols. The fact that quantum correlations such as entanglement and dissonance are required to discriminate quantum states, a natural question arises: is coherence sufficient for UQSD and is there any relation between the degree of coherence and the efficiency of discrimination?

In this paper, we answer these questions affirmatively. In particular, we design a method to find the optimal UQSD by controlling the coherence in a protocol that discriminates two nonorthogonal quantum states. In line with this, we compute the amount of coherence for the optimal UQSD and determine whether this optimality is achieved by the generated coherence in some circumstances.

In our study, we consider a qudit system S that is randomly prepared in one of the d nonorthogonal but linearly independent pure quantum states. The system S is coupled to a $(d+1)$-dimensional auxiliary system A by a joint unitary operator U_{SA}. We give a protocol to construct the U_{SA} for $d \geq 2$. We find that the quantum states post the joint unitary operation do not contain any quantum correlation such as entanglement or quantum discord between the system S and the auxiliary system A. However, quantum coherence is always generated in the auxiliary system A except when the quantum states to be discriminated are mutually orthogonal. The joint unitary thus converts nonorthogonality on the original system S into coherence on the auxiliary system A, and this coherence can be consumed for the discrimination of nonorthogonal states.

2 UQSD with coherence

Quantum coherence [23, 24] is defined with respect to a fixed orthonormal basis $\{|i\rangle\}$ of a system represented by a Hilbert space $\mathcal{H}$. The set of "incoherent" or free states is conceptualized as a set of perfectly distinguishable pure states and their mixtures. Precisely, it is defined by $\mathcal{I} = \big\{\sigma = \sum_i p_i|i\rangle\langle i| : p_i \geq 0, \sum_i p_i = 1\big\}$. The "incoherent" or free operations keep the free states within the set of free states. Precisely, they are completely positive maps, Φ, given by $\Phi(\sigma) = \sum_k E_k \sigma E_k^\dagger$, for a set of incoherent Kraus operators, $\{E_k\}$, so that $\Phi(\sigma) \subseteq \mathcal{I}$ for all $\sigma \subseteq \mathcal{I}$. A measure of coherence (with respect to the von Neumann measurement $\Pi = \{\Pi_i = |i\rangle\langle i|\}$), $C(\rho|\Pi)$, satisfies

(C1) $C(\rho|\Pi) \geq 0$ with equality if and only if $\rho \in \mathcal{I}$,

(C2) $C(\rho|\Pi)$ is nonincreasing under incoherent operations, i.e., $C(\rho|\Pi) \geq C(\Phi(\rho)|\Pi)$ with $\Phi(\mathcal{I}) \subseteq \mathcal{I}$,

(C3) $C(\rho|\Pi)$ is convex in ρ.

There are many important coherence measures [23, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37]. In this paper, we will use two coherence measures. The first coherence measure is an improved version of K coherence [25] based on the Wigner-Yanase skew information $I(\sigma, K) = -\frac{1}{2}\operatorname{tr}([\sqrt{\sigma}, K]^2)$, proposed by Luo *et al.* and defined as [26]

$$C_I(\rho|\Pi) = \sum_i I(\rho, \Pi_i), \tag{1}$$

where $I(\sigma, \Pi_i) = -\frac{1}{2}\operatorname{tr}([\sqrt{\sigma}, \Pi_i]^2)$.

For pure states $|\psi\rangle = \sum_i \psi_i\,|i\rangle$, this measure is equivalent to the coherence measures such as l_2 norm of coherence C_{l_2} and fidelity of coherence C_f [23, 29]:

$$\begin{aligned} C_I(|\psi\rangle\langle\psi||\Pi) &= \sum_{i,j,i\neq j} |\psi_i|^2|\psi_j|^2 = C_f(|\psi\rangle\langle\psi||\Pi) \\ &= C_{l_2}(|\psi\rangle\langle\psi||\Pi). \end{aligned}$$

The second coherence measure C can be either robustness of coherence C_R or l_1 norm of coherence C_{l_1} because these measures have the same expression for pure states [23, 34].

The axiomatic formulation of the coherence measures paves the way for using any measure without significant digressions in the physics content. Quantum coherence has been detected experimentally [25, 38, 39, 40]. Further interesting developments in quantum coherence theory can be explored in Refs. [36, 37, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53].

In UQSD, one seeks for the best quantum measurement to discriminate between the nonorthogonal states $|\phi_i\rangle \in \mathcal{H}$ of the ensemble $\{p_i, |\phi_i\rangle\}_{i=1}^d$ with the least possible "error". An upper bound on the success probability (P_s) of UQSD is given by [6]

$$P_s \leq 1 - \frac{1}{d-1}\sum_{i,j\neq i}\sqrt{p_i p_j}|\langle\phi_i|\phi_j\rangle|. \tag{2}$$

This has an operational meaning in the context of duality between the quantum coherence and the path distinguishability [28].

Let us consider a qudit that is randomly prepared in one of the d nonorthogonal but linearly independent quantum states $|\phi_i\rangle$ in quantum system S, $i = 1, 2, ..., d$, with probabilities p_i. The system

S is coupled to a $(d+1)$-dimensional auxiliary system A by a joint unitary operator U_{SA} such that

$$U_{SA}\left|\phi_i\right\rangle\left|0\right\rangle_A = \sqrt{1-|\alpha_i|^2}\left|\varphi_i\right\rangle\left|i\right\rangle_A + \alpha_i\left|\varphi_i\right\rangle\left|0\right\rangle_A, \tag{3}$$

where $\alpha_i^*\alpha_j\langle\varphi_i|\varphi_j\rangle = \langle\phi_i|\phi_j\rangle$ for $i \neq j$. A protocol for constructing the U_{SA} for $d \geq 2$ is discussed in the Appendix A. After the joint unitary operation U_{SA}, the average quantum state is given as a mixed state $\rho = \sum_{i=1}^d p_i\rho_i = \sum_{i=1}^d p_i\left|\varphi_i\right\rangle\left\langle\varphi_i\right| \otimes \rho_i^A$, where $\rho_i = U_{SA}(\left|\phi_i\right\rangle\left\langle\phi_i\right| \otimes \left|0\right\rangle_A\left\langle 0\right|)U_{SA}^\dagger$ and $\rho_i^A = (1-|\alpha_i|^2)\left|i\right\rangle_A\left\langle i\right| + |\alpha_i|^2\left|0\right\rangle_A\left\langle 0\right| + \sqrt{1-|\alpha_i|^2}\big(\alpha_i\left|0\right\rangle_A\left\langle i\right| + \alpha_i^*\left|i\right\rangle_A\left\langle 0\right|\big)$. Note that ρ_i^A is pure for each i. If we perform the local measurement $M = \{\left|j\right\rangle_A\left\langle j\right|\}_{j=0}^d$ on the auxiliary system, the success probability to discriminate the state is given by

$$P_s = 1 - \mathrm{tr}(\mathbb{I}\otimes\left|0\right\rangle_A\left\langle 0\right|\rho) = \sum_{i=1}^d p_i(1-|\alpha_i|^2), \tag{4}$$

where $\mathbb{I}$ is the unit operator for the system S. Also, since ρ_i^A are pure for all i, the quantum states post the unitary operation do not contain any quantum correlation such as entanglement or quantum discord between the system S and the auxiliary system A. This process only generates and consumes quantum coherence in the auxiliary system A.

Now, we compute the mean of coherence in the basis $\{\left|j\right\rangle_A\}_{i=0}^d$ of the auxiliary system using the measure of coherence defined in equation (1) with the measurement $\Pi^A = \{\Pi_j^A = \left|j\right\rangle_A\left\langle j\right|\}$. We define the mean of coherence as $C_{mean} := \sum_i p_i C_I(\rho_i^A|\Pi^A) = \sum_{i=1}^d p_i\Big[\sum_{j=0}^d I(\rho_i^A, \Pi_j^A)\Big]$ which reduces to

$$C_{mean} = 2\sum_{i=1}^d p_i|\alpha_i|^2\big(1-|\alpha_i|^2\big), \tag{5}$$

and $\tilde{C}_{mean} := \sum_{i=1}^d p_i C(\rho_i^A|\Pi^A)$ which reduces to

$$\tilde{C}_{mean} := 2\sum_{i=1}^d p_i|\alpha_i|\sqrt{1-|\alpha_i|^2}, \tag{6}$$

where C can be either robustness of coherence C_R or l_1 norm of coherence C_{l_1}.

This shows that the success probability is lower bounded by the quantum coherence generated in the auxiliary system, i.e., we have $P_s \geq \frac{1}{2}C_{mean}$. Another important observation here is that coherence is always generated except when the quantum states to be discriminated are mutually orthogonal (see Fig.1).

The joint unitary thus converts nonorthogonality on the original system S into coherence on the auxiliary system A, and this coherence can be consumed for the discrimination of nonorthogonal states (see Ref. [54]).

Also from the point of view of each i, not the mean of coherence, Eqs. (5) and (10) provide us with a heretical relationship between the probability of success $1-|\alpha_i|^2$ and the generated coherence $|\alpha_i|^2\big(1-|\alpha_i|^2\big)$ for each i (see Fig. 2).

Let us assume that the quantum states $\{|\phi_i\rangle\}_{i=1}^d$ satisfy the condition $|\langle\phi_i|\phi_j\rangle| \geq \frac{1}{\sqrt{2}}$ for all $i \neq j$, then we have $|\alpha_i|^2 \geq \frac{1}{2}$ for all i, because $|\alpha_i|^2|\alpha_j|^2 \geq |\alpha_i|^2|\alpha_j|^2|\langle\varphi_i|\varphi_j\rangle|^2 = |\langle\phi_i|\phi_j\rangle|^2 \geq \frac{1}{2}$. In this case, we see from Fig. 2 that $1-|\alpha_i|^2$ decreases when $|\alpha_i|^2\big(1-|\alpha_i|^2\big)$ decreases. This means that

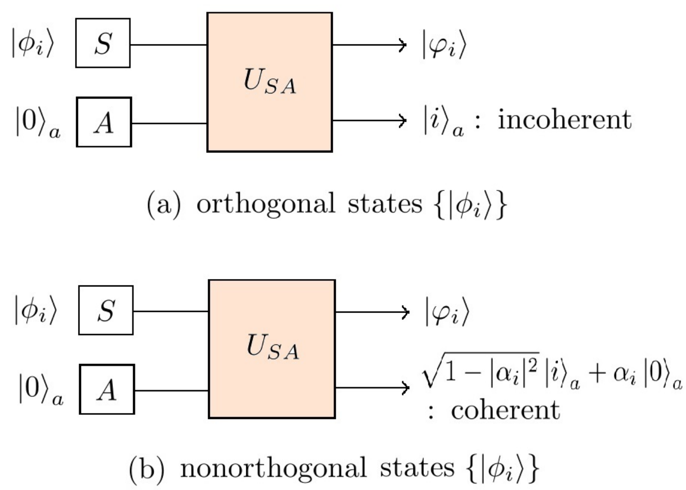

Fig. 1. (a) The UQSD strategy for orthogonal quantum states does not require any coherence. (b) On the contrary, coherence is essential for the UQSD strategy in the case of nonorthogonal quantum states. The degree of nonorthogonality between the quantum states is closely related to the degree of the generated coherence.

if the coherence of i-th quantum state after the joint unitary operation is decreased, then the success probability for result i is also decreased. Conversely, if $|\langle\phi_i|\phi_j\rangle|$ is small enough for all $i \neq j$ and $|\alpha_i|^2$ is not greater than $\frac{1}{2}$, then we can increase the probability of success for the result i by adjusting the i-th coherence to be sufficiently small, as seen in Fig. 2. However, this is possible only with independent relationship for each result i, and it is difficult to find a numerical relationship with the optimal UQSD average above.

3 Mean of coherence for optimal unambiguous discriminations

Here we consider in detail the two-dimensional case. Recall the UQSD protocol in Eq. (3) for $d = 2$. Because it is always possible to make the nonorthogonal quantum states $|\varphi_1\rangle, |\varphi_2\rangle$ in Eq. (3) the same (see Appendix A), we have $\alpha_1^*\alpha_2 = \langle\phi_1|\phi_2\rangle \equiv \gamma$.

If $p_1 = p_2 = \frac{1}{2}$, we have an interesting fact that the extreme values of the success probability P_s and the mean of coherence C_{mean} (or $\tilde{C}_{mean}^{opt}$) are obtained at the same point $|\alpha_1|^2 = |\alpha_2|^2 = \gamma$; see Appendix for the detailed calculation. It follows that we can implement the optimal UQSD strategy by adjusting the mean of coherence to the maximum value in a defined interval $|\gamma|^2 \leq |\alpha_2|^2 \leq 1$ when $|\gamma| \geq \frac{1}{4}$ (see the red lines of (b), (c) and (d) in Fig. 3). Conversely, when $|\gamma| < \frac{1}{4}$, we can implement the optimal discrimination by adjusting the mean of coherence to the local minimum value (see the red line in Fig. 3(a)). Furthermore, the same behaviour is observed for $\tilde{C}_{mean}$ (see the green lines of Fig. 3). Hence, the mean of coherence for the optimal UQSD reduces to $C_{mean}^{opt} \equiv 2|\gamma|\big(1-|\gamma|\big)$ [Eq. (5)] and $\tilde{C}_{mean}^{opt} \equiv 2\sqrt{|\gamma|\big(1-|\gamma|\big)}$ [Eq. (6)] because it has the highest probability of success at $|\gamma| = |\alpha_1|^2 = |\alpha_2|^2$. Note that C_{mean}^{opt} (or $\tilde{C}_{mean}^{opt}$) is the value of mean coherence for the optimal UQSD protocol. Thus, a discrimination strategy or protocol will be an optimal UQSD if the value of mean coherence equals C_{mean}^{opt} (or $\tilde{C}_{mean}^{opt}$).

If $p_1 \neq p_2$, we can measure the coherence for each result i and compare it to $2\sqrt{\frac{p_2}{p_1}}|\gamma|(1-\sqrt{\frac{p_2}{p_1}}|\gamma|)$

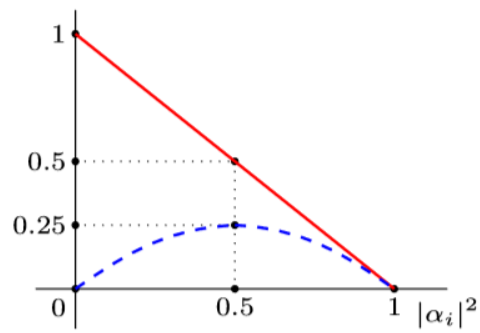

Fig. 2. (Color online) The solid red line is the graph of $1 - |\alpha_i|^2$ and the blue dashed line is the graph of $|\alpha_i|^2(1 - |\alpha_i|^2)$.

for $i = 1$ and $2\sqrt{\frac{p_1}{p_2}}|\gamma|(1-\sqrt{\frac{p_1}{p_2}}|\gamma|)$ for $i = 2$ to determine the optimality of UQSD (see Appendix B). If the measured values of coherence equal the values above, one can implement the optimal strategy to discriminate the given quantum states.

The above discussion can also be extended to general d described in Eq. (3). As in Eq. (2), with $\gamma_{ij} = \langle\phi_i|\phi_j\rangle$, the upper bound for the success probability of the UQSD is given by $P_s \leq 1 - \frac{1}{d-1}\sum_{i,j\neq i}\sqrt{p_i p_j}|\gamma_{ij}|$. This inequality can be saturated provided $p_1|\alpha_1|^2 = p_2|\alpha_2|^2 = \cdots = p_d|\alpha_d|^2$, because $|\alpha_i|^2 = \sqrt{\frac{p_j}{p_i}}|\gamma_{ij}|$ for all $i \neq j$, therefore,

$$\begin{aligned} P_s &= 1 - \sum_i p_i|\alpha_i|^2 = 1 - \sum_i \Big[\frac{1}{d-1}\sum_{j\neq i} p_i\sqrt{\frac{p_j}{p_i}}|\gamma_{ij}|\Big] \\ &= 1 - \frac{1}{d-1}\sum_{i,j\neq i}\sqrt{p_i p_j}|\gamma_{ij}|. \end{aligned}$$

Thus, for any $i \in \{1, 2, ..., d\}$, for example, $i = 1$, we have $d\, p_1|\alpha_1|^2 = \frac{1}{d-1}\sum_{i,j\neq i}\sqrt{p_i p_j}|\gamma_{ij}| \equiv B$. Therefore, the mean of coherence is

$$C_{mean} = 2B\Big(1 - \frac{B}{d^2}\sum_i \frac{1}{p_i}\Big). \tag{7}$$

However, since this upper bound of success probability is not always achievable, it cannot in general be regarded as an optimal success probability. Likewise, we cannot be certain that the mean of coherence in Eq. (7) is for an optimal discrimination. It is only possible to estimate how similar or close our UQSD is to the optimal UQSD by comparing the computed mean value with that in Eq. (7). However, when the quantum states $\{|\phi_i\rangle\}$ satisfy the following two conditions, we can obtain the optimal result.

Condition 1. $\frac{|\gamma_{ij}||\gamma_{ik}|}{|\gamma_{jk}|} = \frac{|\gamma_{il}||\gamma_{im}|}{|\gamma_{lm}|}$ for unequal i, j, k, l, m. This makes it possible for all $|\varphi_i\rangle$ to be equal in Eq. (3), i.e., $U_{SA}|\phi_i\rangle|0\rangle_A = \sqrt{1-|\alpha_i|^2}\,|\varphi\rangle|i\rangle_A + \alpha_i|\varphi\rangle|0\rangle_A$, where $\alpha_i^*\alpha_j = \langle\phi_i|\phi_j\rangle$. Then $\frac{|\gamma_{1j}||\gamma_{1k}|}{|\gamma_{jk}|} = |\alpha_1|^2$ for any $j \neq k$.

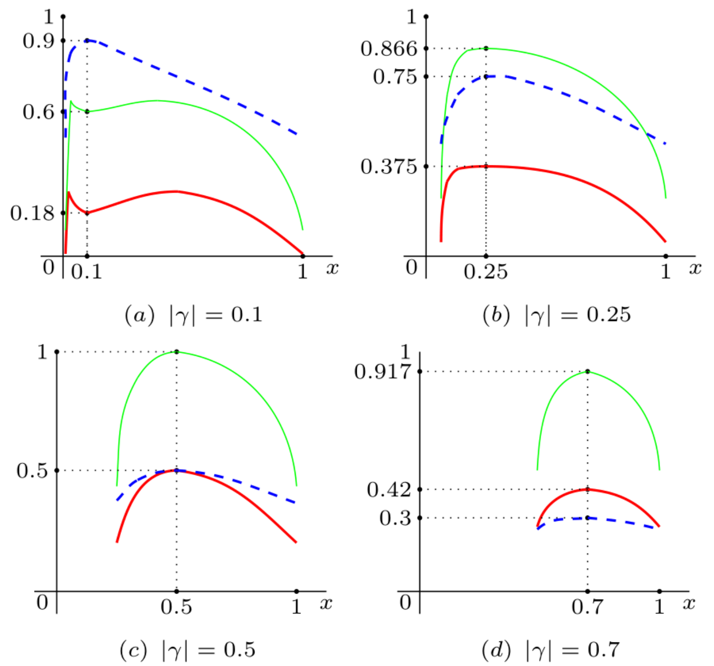

Fig. 3. (Color online) Plots of $P_s|_{x=|\alpha_1|^2}$ (dashed blue), $C_{mean}|_{x=|\alpha_1|^2}$ (solid red) and $\tilde{C}_{mean}|_{x=|\alpha_1|^2}$ (thin green), when $p_1 = p_2 = \frac{1}{2}$, against x for different values of $|\gamma|$.

Condition 2. $p_i|\gamma_{ki}|^2 = p_j|\gamma_{kj}|^2$ for unequal i, j, k. This allows us to design a strategy which satisfies $p_1|\alpha_1|^2 = p_2|\alpha_2|^2 = \cdots = p_d|\alpha_d|^2$.

Therefore, when the above two conditions are satisfied, we can verify that the UQSD is optimal by comparing the mean value of the measured coherence with Eq. (7).

4 The protocol with noise

Next, we try to understand how the UQSD protocol using quantum coherence is affected when the input state is subject to noise. This noise can be modelled in a variety of ways, and depends on the actual implementation of the relevant devices. In the literature, arguably the most popular theoretical model of noise is admixture with white noise. But as in our protocol of UQSD there is already a bias in the input state, it is plausible that the environmental noise will thereby be biased as well. We restrict ourselves, in the noisy scenario, to the case where there are two inputs to the distinguishing device, and they are respectively $|0\rangle$ and $|+\rangle$, where $\langle +|0\rangle = 1/\sqrt{2}$. We assume the noise model where the density matrices corresponding to the states $|0\rangle, |+\rangle$ become $\rho_0 = p|0\rangle\langle 0| + \frac{1-p}{2}\tilde{\mathbb{I}}_2$ and $\rho_+ = p|+\rangle\langle +| + \frac{1-p}{2}\tilde{\mathbb{I}}_2$, where $\tilde{\mathbb{I}}_2 = |0\rangle\langle 0| + |+\rangle\langle +|$. Calculating the final state after the unitary transformation, we see that no entanglement or discord is generated. And, for various values of the noise parameter $(1-p)$, we have calculated the value of quantum coherence. Please refer to the Appendix C for the detailed analysis. We also explicitly show that the reliability of the distinguishing protocol decreases from 1 to $\frac{1+p}{2}$ in the presence of noise, where $(1-p)$ is the strength of the noise.

5 Conclusion

Identifying resources for quantum state discrimination is of fundamental importance. Use of quantum correlations as a resource for the same has been studied extensively. In this paper, we have investigated the role of quantum coherence in unambiguously discriminating nonorthogonal but linearly independent pure quantum states, assisted by an auxiliary system. We provide a relationship between the success probability of the discriminating strategy and the mean coherence generated on the auxiliary system for several important coherence measures. The degree of the generated coherence depends on the nonorthogonality between the input quantum states. We can effectively use the mean of coherence to improve the efficiency of the strategy for each individual result of the performed measurement. Finally, we compute the coherence that is generated when an optimal unambiguous discrimination strategy is implemented in some situations. In these cases, we can use the mean of coherence to determine whether the discrimination strategy is optimal or not. In particular, for unambiguous discrimination between two pure qubit states, we show that the receiver can obtain the optimal strategy by controlling the mean coherence to the maximum or minimum value without feedback from the sender. Our result will open up new investigations in the use of coherence in quantum state discrimination.

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6 Appendix

6.1 The joint unitary operators for $d \geq 2$ *quantum states*

We construct a unitary operator U_{SA} in the UQSD strategy that discriminates between two quantum states $|\phi_1\rangle$ and $|\phi_2\rangle$. Let $\langle\phi_1|\phi_2\rangle = \gamma$ and $\{|i\rangle_A\}_{i=0}^2$ be an orthonormal basis of the auxiliary system A. We assume that the system S is 2 dimensional. Take a vector $|\phi_1^+\rangle \in S$ such that $\langle\phi_1|\phi_1^+\rangle = 0$ and $|\phi_2\rangle = \gamma|\phi_1\rangle + \sqrt{1-|\gamma|^2}\,|\phi_1^+\rangle$. Then $\{|\phi_1\rangle|i\rangle_A, |\phi_1^+\rangle|i\rangle_A\}_i$ is an orthonormal basis of the whole system SA. Let $0 < |\alpha| \leq 1$, $|\upsilon_{\alpha,1}\rangle_A = \sqrt{1-|\alpha|^2}\,|1\rangle_A + \alpha|0\rangle_A$ and $|\upsilon_{\alpha,2}\rangle_A = \frac{\gamma(1-|\alpha|^2)}{\alpha^*\sqrt{1-|\gamma|^2}}|0\rangle_A -$

$\frac{\gamma\sqrt{1-|\alpha|^2}}{\sqrt{1-|\gamma|^2}}|1\rangle_A + \frac{\sqrt{|\alpha|^2-|\gamma|^2}}{|\alpha|\sqrt{1-|\gamma|^2}}|2\rangle_A$. Then it is easy to see that $\langle\upsilon_{\alpha,1}|\upsilon_{\alpha,2}\rangle_A = 0$. Moreover, let us take a unit vector $|\upsilon_{\alpha,0}\rangle_A \in A$ such that $\{|\upsilon_{\alpha,j}\rangle_A\}_{j=0}^2$ is an orthonormal basis of the auxiliary system A, and take two unit vectors $|\varphi\rangle, |\varphi^+\rangle$ in S such that $\langle\varphi|\varphi^+\rangle = 0$. Let $U_{\gamma,\alpha}$ be a transform from the orthonormal basis $\{|\phi_1\rangle|i\rangle_A, |\phi_1^+\rangle|i\rangle_A\}_i$ to the orthonormal basis $\{|\varphi\rangle|\upsilon_{\alpha,j}\rangle_A, |\varphi^+\rangle|\upsilon_{\alpha,j}\rangle_A\}_j$ that satisfies $U_{\gamma,\alpha}|\phi_1\rangle|0\rangle_A = |\varphi\rangle|\upsilon_{\alpha,1}\rangle_A$ and $U_{\gamma,\alpha}|\phi_1^+\rangle|0\rangle_A = |\varphi\rangle|\upsilon_{\alpha,2}\rangle_A$. Then $U_{\gamma,\alpha}$ is a unitary transformation on the system SA such that:

$$\begin{aligned} U_{\gamma,\alpha}|\phi_1\rangle|0\rangle_A &= \sqrt{1-|\alpha|^2}|\varphi\rangle|1\rangle_A + \alpha|\varphi\rangle|0\rangle_A, \\ U_{\gamma,\alpha}|\phi_2\rangle|0\rangle_A &= \sqrt{1-\frac{|\gamma|^2}{|\alpha|^2}}|\varphi\rangle|2\rangle_A + \frac{\gamma}{\alpha^*}|\varphi\rangle|0\rangle_A. \end{aligned}$$

Next, let us find out about the case of $d \geq 3$. For d linearly independent quantum states $|\phi_i\rangle$ with $\langle\phi_i|\phi_j\rangle = \gamma_{ij}$, the unitary operator U_{SA} of (3) can be achieved by expanding and repeating similar tasks as above, but it must include more complex process. We first find the following orthonormal basis $\{|\phi_i'\rangle\}$ of the system S sequentially from the states $|\phi_i\rangle$:

$$|\phi_1'\rangle = |\phi_1\rangle \quad \text{and} \quad |\phi_i'\rangle = \frac{|\phi_i\rangle - \sum_{j=1}^{i-1}\gamma_{ji}'|\phi_j'\rangle}{\sqrt{1-\sum_{j=1}^{i-1}|\gamma_{ji}'|^2}} \tag{8}$$

with $\langle\phi_j'|\phi_i\rangle = \gamma_{ji}'$ for $2 \leq i \leq d$. For this we also know that $|\phi_i\rangle$ can be represented as a combination of $\{|\phi_i'\rangle\}$, *i.e.*,

$$|\phi_i\rangle = \sum_{j=1}^{i-1}\gamma_{ji}'|\phi_j'\rangle + \sqrt{1-\sum_{j=1}^{i-1}|\gamma_{ji}'|^2}\,|\phi_i'\rangle \tag{9}$$

for $2 \leq i \leq d$. Our aim here is to find states $|\varphi_i'\rangle$ and $|\upsilon_{\alpha_i}\rangle_A$ with $\langle\upsilon_{\alpha_i}|\upsilon_{\alpha_j}\rangle_A = 0$ for $i \neq j$ that satisfy the Eq. (3) when $U_{SA}|\phi_i'\rangle|0\rangle_A = |\varphi_i'\rangle|\upsilon_{\alpha_i}\rangle_A$. These states can be found sequentially, starting with $|\varphi_1'\rangle = |\varphi_1\rangle$ and $|\upsilon_{\alpha_1}\rangle = \sqrt{1-|\alpha_1|^2}|1\rangle + \alpha_1|0\rangle$. Of course, $\alpha_i^*\alpha_j\langle\varphi_i|\varphi_j\rangle = \langle\phi_i|\phi_j\rangle$ means that the inner products between states before and after the unitary U_{SA} are preserved, and therefore the existence of U_{SA} satisfying the Eq. (3) is guaranteed.

In addition, if α_i are satisfied with $\alpha_i^*\alpha_j = \langle\phi_i|\phi_j\rangle$ for $i \neq j$, we can find U_{SA} in a simpler way. Here we also use the orthonormal basis $\{|\phi_i'\rangle\}$ of the system S in (8), and find quantum states $|\upsilon_{\alpha_i}\rangle$ with $\langle\upsilon_{\alpha_i}|\upsilon_{\alpha_j}\rangle = 0$ for $(i \neq j)$ satisfying that

$$\sum_{j=1}^{i-1}\gamma_{ji}'|\upsilon_{\alpha_j}\rangle + \sqrt{1-\sum_{j=1}^{i-1}|\gamma_{ji}'|^2}\,|\upsilon_{\alpha_i}\rangle = \sqrt{1-|\alpha_i|^2}|i\rangle + \alpha_i|0\rangle.$$

In the above equation, the part before the equal sign is the same as the form for $|\phi_i'\rangle$ in (9) and thereby we can find the states $|\upsilon_{\alpha_i}\rangle$ in sequence, starting with $|\upsilon_{\alpha_1}\rangle = \sqrt{1-|\alpha_1|^2}|1\rangle + \alpha_1|0\rangle$. Then the joint unitary operator U_{SA} that result in $U_{SA}|\phi_i'\rangle|0\rangle_A = |\varphi\rangle|\upsilon_{\alpha_i}\rangle_A$ for any state $|\varphi\rangle$ satisfy Eq. (3).

6.2 Relation of P_s, C_{mean} and $\tilde{C}_{mean}$ for two quantum states

The success probability to discriminate between two quantum states $|\phi_1\rangle$ and $|\phi_2\rangle$ is

$$P_s = 1 - \mathrm{tr}(\mathbb{I}\otimes|0\rangle_A\langle 0|\rho) = \sum_{i=1}^{2}p_i(1-|\alpha_i|^2). \tag{10}$$

Note that for the U_{SA} in the above section, we have $|\alpha_1|^2|\alpha_2|^2 = |\gamma|^2$. Denoting $x = |\alpha_1|^2$, $P_s(x) = p_1(1-x) + p_2\big(1 - \frac{|\gamma|^2}{x}\big)$. For the optimal success probability, we require $P_s'(x) = -p_1 + p_2\frac{|\gamma|^2}{x^2} = 0$. This yields $p_1|\alpha_1|^2 = p_2|\alpha_2|^2$. That is, if $|\alpha_1|^2 = \sqrt{\frac{p_2}{p_1}}|\gamma|$, then we can distinguish $|\phi_1\rangle$ and $|\phi_2\rangle$ with the optimal success probability

$$P_s^{opt} = 1 - 2p_1|\alpha_1|^2 = 1 - 2\sqrt{p_1p_2}|\gamma|.$$

Since $p_1p_2|\gamma|^2 = p_1^2|\alpha_1|^4 = p_2^2|\alpha_2|^4$, the mean of coherence, $C_{mean} = 2\sum_{i=1}^d p_i|\alpha_i|^2\big(1-|\alpha_i|^2\big)$, for the optimal UQSD is

$$C_{mean}^{opt} = 2|\gamma|(2\sqrt{p_1p_2} - |\gamma|).$$

When $p_1 = p_2 = \frac{1}{2}$, we have

$$C_{mean}(x) = \big[x(1-x) + \frac{|\gamma|^2}{x}\big(1 - \frac{|\gamma|^2}{x}\big)\big],$$

and its first-order derivative with respect to x

$$C_{mean}'(x) = -\frac{1}{x^3}(x-|\gamma|)(x+|\gamma|)(2x^2 - x + 2|\gamma|^2).$$

Thus $C_{mean}'(x) = 0$ has three roots: $x_1 = \gamma$, $x_2 = \frac{1+\sqrt{1-16|\gamma|}}{4}$, and $x_3 = \frac{1+\sqrt{1-16|\gamma|}}{4}$, where $|\gamma| \leq \frac{1}{4}$.

Moreover, from the second-order derivative of $C_{mean}(x)$ with respect to x

$$C_{mean}''(x) = -2 + 2\frac{|\gamma|^2}{x^3} - 6\frac{|\gamma|^4}{x^4},$$

it follows that

$$C_{mean}''(x)\Big|_{|\gamma|=x} \begin{cases} \leq 0 & \text{when } |\gamma| \geq \frac{1}{4}, \\ > 0 & \text{when } |\gamma| < \frac{1}{4}. \end{cases}$$

Therefore, there is only one extreme point at $x = |\gamma| = |\alpha_1|^2$ when $|\gamma| \geq \frac{1}{4}$.

Furthermore, for the second-type mean of coherence $\tilde{C}_{mean}(x)$ when $p_1 = p_2 = \frac{1}{2}$, we have

$$\tilde{C}_{mean}(x) = \sqrt{x - x^2} + |\gamma|\sqrt{\frac{1}{x} - \frac{|\gamma|^2}{x^2}}$$

and

$$\tilde{C}_{mean}'(x) = \frac{1}{2}\Big[\frac{1-2x}{\sqrt{x-x^2}} - \frac{|\gamma|(x - 2|\gamma|^2)}{x^2\sqrt{x - |\gamma|^2}}\Big].$$

Thus, $\tilde{C}_{mean}$ also has an extreme value at $x = |\gamma| = |\alpha_1|^2$.

6.3 UQSD using quantum coherence in presence of noise

Suppose that the states to be distinguished using UQSD are $|0\rangle$ and $|+\rangle$. We consider the noise model where the density matrices corresponding to the states $|0\rangle, |+\rangle$, become

$$\rho_0 = p|0\rangle\langle 0| + \frac{1-p}{2}\tilde{\mathbb{I}}_2, \quad \text{and} \quad \rho_+ = p|+\rangle\langle +| + \frac{1-p}{2}\tilde{\mathbb{I}}_2, \tag{11}$$

where $\tilde{\mathbb{I}}_2 = |0\rangle\langle 0| + |+\rangle\langle +|$. It is possible to calculate the degree of quantum coherence of a quantum state if its spectral decomposition is known. Therefore, we first obtain the spectral decomposition of the above quantum states for any noise $1-p$ $(0 \le p \le 1)$ as follows:

$$\rho_0 = q|\psi_+\rangle\langle\psi_+| + (1-q)|\psi_-\rangle\langle\psi_-| \quad \text{and} \quad \rho_+ = q|\phi_+\rangle\langle\phi_+| + (1-q)|\phi_-\rangle\langle\phi_-|,$$

where

$$\begin{aligned} |\psi_+\rangle &= a_0\,|0\rangle + \sqrt{1-a_0^2}\,|1\rangle\,, \;\; |\psi_-\rangle = \sqrt{1-a_0^2}\,|0\rangle - a_0\,|1\rangle\,, \\ |\phi_+\rangle &= a_+\,|0\rangle + \sqrt{1-a_+^2}\,|1\rangle\,, |\phi_-\rangle = \sqrt{1-a_+^2}\,|0\rangle - a_+\,|1\rangle \end{aligned}$$

with $a_0^2 = \frac{1}{2} + \frac{\sqrt{2}(1+p)}{4\sqrt{1+p^2}}, a_+^2 = \frac{1}{2} + \frac{\sqrt{2}(1-p)}{4\sqrt{1+p^2}}$ and $q = \frac{1}{2} + \frac{\sqrt{2}\sqrt{1+p^2}}{4}$. If U_{SA} is a joint unitary transformation on the system SA such that:

$$\begin{aligned} |\Phi_0\rangle = U_{SA}\,|0\rangle\,|0\rangle_A &= \sqrt{1-\alpha^2}\,|\varphi_0\rangle\,|1\rangle_A + \alpha\,|\varphi_0\rangle\,|0\rangle_A\,, \\ |\Phi_+\rangle = U_{SA}\,|+\rangle\,|0\rangle_A &= \sqrt{1-\frac{\gamma^2}{\alpha^2}}\,|\varphi_+\rangle\,|2\rangle_A + \frac{\gamma}{\alpha}\,|\varphi_+\rangle\,|0\rangle_A\,, \end{aligned}$$

where $\gamma = \langle 0|+\rangle = \frac{\sqrt{2}}{2}$, then the action of the unitary operator U_{SA} on the states ρ_0 and ρ_+ is

$$U_{SA}(\rho_0 \otimes |0\rangle_A\langle 0|)U_{SA}^\dagger = \frac{(1+p)}{2}|\Phi_0\rangle\langle\Phi_0| + \frac{1-p}{2}|\Phi_+\rangle\langle\Phi_+|, \tag{12}$$

$$U_{SA}(\rho_+ \otimes |0\rangle_A\langle 0|)U_{SA}^\dagger = \frac{(1+p)}{2}|\Phi_+\rangle\langle\Phi_+| + \frac{1-p}{2}|\Phi_0\rangle\langle\Phi_0|, \tag{13}$$

respectively. The unitary transformations in Eq. (12) and Eq. (13) can be rewritten as

$$\begin{aligned} U_{SA}(\rho_i \otimes |0\rangle_A\langle 0|)U_{SA}^\dagger &= |\varphi_i\rangle\langle\varphi_i| \otimes \rho_i^A \\ &= |\varphi_i\rangle\langle\varphi_i| \otimes \Big\{q\,|\Phi_{i,+}\rangle_A\langle\Phi_{i,+}| + (1-q)\,|\Phi_{i,-}\rangle_A\langle\Phi_{i,-}|\Big\}, \end{aligned} \tag{14}$$

where

$$|\Phi_{i,+}\rangle_A = \Big[\big\{\alpha a_i - (\alpha - \frac{1}{\alpha})\sqrt{1-a_i^2}\big\}\,|0\rangle + \sqrt{1-\alpha^2}(a_i - \sqrt{1-a_i^2})\,|1\rangle + \sqrt{(2-\frac{1}{\alpha^2})(1-a_i^2)}\,|2\rangle\Big]_A$$

and

$$|\Phi_{i,-}\rangle_A = \Big[\big\{\alpha(a_i + \sqrt{1-a_i^2}) - \frac{a_i}{\alpha}\big\}\,|0\rangle + \sqrt{1-\alpha^2}(a_i + \sqrt{1-a_i^2})\,|1\rangle - a_i\sqrt{2-\frac{1}{\alpha^2}}\,|2\rangle\Big]_A,$$

for $i = 0, +$. We can see that no quantum entanglement or discord is generated after the unitary transformation for any noise $1-p$.

Next, we calculate quantum coherence with respect to measurement $\Pi = \{|0\rangle_A\langle 0|\,, |1\rangle_A\langle 1|\,, |2\rangle_A\langle 2|\}$ on system A,

$$C_I(\rho_i^A) = \sum_{j=0}^{2}\Big[q\big\{(+)_{i,j}^2 - (+)_{i,j}^4\big\} + (1-q)\big\{(-)_{i,j}^2 - (-)_{i,j}^4\big\} - 2\sqrt{q(1-q)}(+)_{i,j}^2(-)_{i,j}^2\Big],$$

where $(+)_{i,j} = \langle\Phi_{i,+}|j\rangle_A, (-)_{i,j} = \langle\Phi_{i,-}|j\rangle_A$ for $i = 0, +$ and $j = 0, 1, 2$. In addition, with some tedious calculation, we can predict the value of noise $1 - p$ from the measured values of quantum coherence when α is fixed.

For example, when $p = 1$ (the case of no noise), we have

$$C_I(\rho_0^A) = 2\alpha^2(1 - \alpha^2), \quad C_I(\rho_+^A) = \frac{1}{\alpha^2}(1 - \frac{1}{2\alpha^2}).$$

This is the measured value of quantum coherence in the absence of noise. If $\alpha^2 = \gamma = \frac{\sqrt{2}}{2}$, then we have

$$\begin{aligned} p = 1 \quad &: \quad C_I(\rho_0^A) = C_I(\rho_+^A) \approx 0.414, \\ p = 0.5 \quad &: \quad C_I(\rho_0^A) = C_I(\rho_+^A) \approx 0.287, \\ p = 0.2 \quad &: \quad C_I(\rho_0^A) = C_I(\rho_+^A) \approx 0.271, \\ p = 0 \quad &: \quad C_I(\rho_0^A) = C_I(\rho_+^A) \approx 0.269. \end{aligned}$$

Now, if we happen to know that the probability of the measurement outcome is $|0\rangle$, we can calculate the probability with which the input state $|0\rangle$ was sent. We call this probability *reliability* when $|0\rangle$ clicks in measurement M, and denote it by R_0 (see Ref. [22]). The expression for R_0 is given by

$$R_0 = \text{Pr}(|0\rangle \big| |0\rangle) = \frac{\text{Pr}(|0\rangle) \times \text{Pr}(|0\rangle \big| |0\rangle)}{\text{Pr}(|0\rangle) \times \text{Pr}(|0\rangle \big| |0\rangle) + \text{Pr}(|+\rangle) \times \text{Pr}(|0\rangle \big| |+\rangle)}, \tag{15}$$

where $\text{Pr}(|\cdot\rangle)$ denotes the probability that $|\cdot\rangle$ was sent, $\text{Pr}(|*\rangle \,|\, |\circ\rangle)$ denotes the probability that the outcome of M is $|*\rangle$ when $|\circ\rangle$ was sent, and the Bayes rule [55, 56] is used in Eq. (15). Assuming that the states $|0\rangle$ and $|1\rangle$ were chosen with equal probabilities, we get

$$R_0 = \frac{\frac{1}{2} \times p_0}{\frac{1}{2} \times p_0 + \frac{1}{2} \times p_+} = \frac{p_0}{p_0 + p_+}, \tag{16}$$

where

$$\begin{aligned} p_0 &= {}_A\langle 0|\text{Tr}_S\left(\frac{1+p}{2}|\Phi_0\rangle\langle\Phi_0| + \frac{1-p}{2}|\Phi_+\rangle\langle\Phi_+|\right)|0\rangle_A, \\ p_+ &= {}_A\langle 0|\text{Tr}_S\left(\frac{1+p}{2}|\Phi_+\rangle\langle\Phi_+| + \frac{1-p}{2}|\Phi_0\rangle\langle\Phi_0|\right)|0\rangle_A. \end{aligned}$$

Since $\text{Tr}_S|\Phi_0\rangle\langle\Phi_0| = \mathbb{P}[(1 - |\alpha_0|^2)|0\rangle + \alpha_0|2\rangle]$ and $\text{Tr}_S|\Phi_+\rangle\langle\Phi_+| = \mathbb{P}[(1 - |\alpha_+|^2)|1\rangle + \alpha_+|2\rangle]$, where $\mathbb{P}(\cdot)$ denotes the projector of the vector in the argument, we can write

$$p_0 = \frac{1+p}{2}(1 - |\alpha_0|^2), \quad p_+ = \frac{1-p}{2}(1 - |\alpha_0|^2). \tag{17}$$

Substituting these in Eq. (16), we get

$$R_0 = \frac{1+p}{2}. \tag{18}$$

Performing a similar analysis for the reliability R_+, when $|+\rangle$ clicks in M, we obtain

$$R_+ = R_0 = \frac{1+p}{2}, \tag{19}$$

which can be called the "reliability of the entire distinguishing process". Hence, we can say that the reliability of the distinguishing process decreases from 1 to $\frac{1+p}{2}$, when noise acts on the system, where $1 - p$ $(0 \leq p \leq 1)$ is strength of the noise for the noise model under consideration. Note that the reliability is lower bounded by $1/2$.