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CHANNEL FIDELITIES FOR HIGH-FIDELITY APPROACH IN KLM SCHEME

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We study channel fidelity for the high-fidelity approach in the Knill-Laflamme-Milburn (KLM) scheme. We examine an optimal channel fidelity f_{opt} and identify the corresponding KLM ancilla state. In the limit of large n, where $2n$ is the number of the ancilla qubits, we find $f_{opt} = 1 - \frac{\pi^2}{6n^2} + \frac{2\pi^2}{9n^3}$. We see that as n increases f_{opt} approaches to 1 slightly faster than $f = 1 - \frac{2}{n^2}$ which is the channel fidelity computed by Franson et. al. in the limit of large n. We also compute the channel fidelity for the ancilla state that gives a lower bound of success probability of quantum teleportation.

Keywords: channel fidelity, KLM, high fidelity

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1 Introduction

Channel fidelity [1] is one of characters that represent performance of a quantum communication channel. Quantum teleportation[2] takes an important role in quantum communication. Quantum teleportation also produces promising strategy in quantum computation[3, 4]. Photon is easily transmitted far away with scarcely being affected with noise in ordinary temperature. Therefore, photon is one of hopeful media of quantum information. We can quantum teleport a photon only probabilistically[5]. Knill, Laflamme and Milburn(KLM) [6] have invented a scheme to quantum teleport a photon with success probability near to 1 by introducing an adequate $2n$-qubits ancilla state that is called a KLM state. Franson et. al.[7] have proposed an approach to improve the success probability in the sense of fidelity by tuning the KLM ancilla state. Their result, however, depends on the large n analysis, and no particular ancilla state is given concretely.

The purpose of this paper is to study optimal channel fidelity for the high fidelity approach in the KLM scheme. We also give the corresponding ancilla state. The state $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ is the most difficult state to quantum teleport for the high fidelity approach[8]. We identify the ancilla state that maximizes the success probability for $|+\rangle$. We also evaluate the channel fidelity for this ancilla state. We exhibit a simple optical circuit that probabilistically produces an intended ancilla state from the original KLM state[9]. Preparing KLM-type ancilla states has already been discussed in the literature [10, 11].

2 Channel fidelities

We prepare a $2n$-qubits ancilla state as $|t_n\rangle = \Sigma_{i=0}^{n} c(i)|0\rangle^{n-i}|1\rangle^{i}|0\rangle^{i}|1\rangle^{n-i}$, where $|0\rangle^i$ means i photons in the state $|0\rangle$ etc. and $c(i)$'s are real coefficients normalized as $\Sigma_{i=0}^{n} c(i)^2 = 1$. It is convenient to introduce a vector ${}^t\mathbf{c} = {}^t(c(0), c(1), \cdots, c(n+1))$. In the original KLM scheme they are settled as $c(i) = \frac{1}{\sqrt{n+1}}, i = 0, 1, \cdots n$. We consider to teleport a quantum state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle = e^{i\gamma}(\cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle), |\alpha|^2 + |\beta|^2 = 1$. We perform $n+1$-point quantum Fourier transformation $\hat{F}_{n+1}$ on the state $|\psi\rangle$ and the first n qubits in the ancilla state. Suppose we observe $k (0 \le k \le n+1)$ photons after the transformation. When $k = 0$ and $k = n+1$ we loose the original state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ completely. In another case, we obtain the quasi teleported state

$$|q_k\rangle = \frac{\alpha c(k)|0\rangle + \beta c(k-1)|1\rangle}{\sqrt{|\alpha|^2 c(k)^2 + |\beta|^2 c(k-1)^2}} \tag{1}$$

at the k-th qubit in the latter half of the ancilla qubits. To obtain the state $|q_k\rangle$ in the form of Eq.(1), we should perform certain relative phase shift between the states $|0\rangle$ and $|1\rangle$ depending on the observed k-photon state. The probability p_k to obtain the state $|q_k\rangle$ is given by

$$p_k = \Sigma_k |\langle k|\hat{F}_{n+1}|\psi\rangle|t_n\rangle|^2 = |\alpha|^2 c(k)^2 + |\beta|^2 c(k-1)^2, \tag{2}$$

where the summation about k runs over all possible k-photon states and we have used $\Sigma_k |k\rangle\langle k| = \hat{I}_k$ with $\hat{I}_k$ the identity operator on any k-photon state. In the high-fidelity approach the success probability $p_{\mathbf{c}}(|\psi\rangle)$ is defined by the expectation value of the square of the fidelity $|\langle\psi|q_k\rangle|^2, k = 1, 2, \cdots, n$; $p_{\mathbf{c}}(|\psi\rangle)$ is defined as $p_{\mathbf{c}}(|\psi\rangle) = \sum_{k=1}^{n} p_k |\langle\psi|q_k\rangle|^2$. Therefore, the success probability for the state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ is given by

$$p_{\mathbf{c}}(|\psi\rangle) = \Sigma_{k=1}^{n}(|\alpha|^2 c(k) + |\beta|^2 c(k-1))^2. \tag{3}$$

Channel fidelity $f_{\mathbf{c}}$ is defined by $f_{\mathbf{c}} = \int d\psi p_{\mathbf{c}}(|\psi\rangle) = \frac{1}{4\pi}\int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \sin\theta p_{\mathbf{c}}(|\psi\rangle)$ that is an average over all input pure states uniformly distributed on the Bloch sphere surface. Using $\int_0^{\pi}\cos^4\frac{\theta}{2}\sin\theta d\theta = \int_0^{\pi}\sin^4\frac{\theta}{2}\sin\theta d\theta = \frac{2}{3}$ and $\int_0^{\pi}\cos^2\frac{\theta}{2}\sin^2\frac{\theta}{2}\sin\theta d\theta = \frac{1}{3}$, $f_{\mathbf{c}}$ is given by $f_{\mathbf{c}} = {}^t\mathbf{c}\tilde{\mathbf{A}}\mathbf{c}$, where $\tilde{A}$ is the following $(n+1)\times(n+1)$ matrix

$$\tilde{A} = \frac{1}{3}\begin{pmatrix} 1 & \frac{1}{2} & 0 & \dots & 0 & 0 \\ \frac{1}{2} & 2 & \frac{1}{2} & \ddots & 0 & 0 \\ 0 & \frac{1}{2} & 2 & \ddots & 0 & \vdots \\ \vdots & \dots & \ddots & \ddots & \frac{1}{2} & 0 \\ 0 & \dots & \dots & \frac{1}{2} & 2 & \frac{1}{2} \\ 0 & \dots & \dots & 0 & \frac{1}{2} & 1 \end{pmatrix}. \tag{4}$$

Let us μ_n be the largest eigenvalue of $\tilde{A}$ and $\mathbf{u}$ be the corresponding normalized eigenvector. Setting $\mathbf{c}$ to be $\mathbf{u}$, we obtain an optimal fidelity $f_{opt} = f_{\mathbf{u}} = \mu_n$.

The eigenvalues and the eigenvectors of the matrices $\tilde{A}$ have been studied by Yueh[12]. The eigenvector ${}^t\mathbf{u} = {}^t(u(0), u(1), \cdots, u(n))$ is given by $u(j) = u(0)(\sin(j+1)\theta + 2\sin j\theta)/\sin\theta$, where $\theta (> 0)$ is the smallest angle satisfying

$$\frac{1}{4}\sin(n+2)\theta + \sin(n+1)\theta + \sin n\theta = 0. \tag{5}$$

Using θ, the largest eigenvalue $\mu_n$can be written as[12]

$$\mu_n = \frac{2}{3} + \frac{1}{3}\cos\theta. \tag{6}$$

From the normalization condition $u(0)^2$ is given by

$$u(0)^2 = \frac{\sin^2\theta}{\Sigma_{j=0}^n(\sin(j+1)\theta + \sin j\theta)^2}. \tag{7}$$

We introduce the following $(n+1)\times(n+1)$ matrix A[8] that has some nice properties

$$A = \frac{1}{4}\begin{pmatrix} 1 & 1 & 0 & \dots & 0 & 0 \\ 1 & 2 & 1 & \ddots & 0 & 0 \\ 0 & 1 & 2 & \ddots & 0 & \vdots \\ \vdots & \dots & \ddots & \ddots & 1 & 0 \\ 0 & \dots & \dots & 1 & 2 & 1 \\ 0 & \dots & \dots & 0 & 1 & 1 \end{pmatrix}. \tag{8}$$

The largest eigenvalue of A is given by $\lambda_n = \frac{1}{2} + \frac{1}{2}\cos\frac{\pi}{n+1}$ and the corresponding eigenvectors is denoted as ${}^t\mathbf{v} = {}^t(v(0), v(1), \cdots, v(n))$, where $v(j) = v(0)(\sin(j+1)\frac{\pi}{n+1} + \sin j\frac{\pi}{n+1})/\sin\frac{\pi}{n+1}$. From the normalization condition $v(0)^2$ is given by

$$v(0)^2 = \frac{\sin^2\frac{\pi}{n+1}}{\Sigma_{j=0}^n(\sin(j+1)\frac{\pi}{n+1} + \sin j\frac{\pi}{n+1})^2}. \tag{9}$$

The two matrices $\tilde{A}$ and A are related as

$$\tilde{A} = \frac{2}{3}A + \frac{1}{3}E - \frac{1}{6}\Gamma, \tag{10}$$

where E is the $(n+1)\times(n+1)$ identity matrix and $(n+1)\times(n+1)$ matrix Γ is defined by

$$\Gamma = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & \vdots \\ \vdots & \dots & \ddots & 0 \\ 0 & \dots & 0 & 1 \end{pmatrix}. \tag{11}$$

From the inequality $f_{opt} = {}^t\mathbf{u}\tilde{A}\mathbf{u} \geq {}^t\mathbf{v}\tilde{A}\mathbf{v} = f_{\mathbf{v}}$ we have the following inequality

$$f_{opt} \geq f_{\mathbf{v}} = \frac{2}{3} + \frac{1}{3}\cos\frac{\pi}{n+1} - \frac{1}{3}v(0)^2, \tag{12}$$

where we have used $v(0) = v(n)$. From ${}^t\mathbf{u}A\mathbf{u} \leq {}^t\mathbf{v}A\mathbf{v}$ we have another inequality $f_{opt} = {}^t\mathbf{u}\tilde{A}\mathbf{u} = \frac{2}{3}{}^t\mathbf{u}A\mathbf{u} + \frac{1}{3} - \frac{1}{6}(u(0)^2 + u(n)^2) \leq \frac{2}{3}\lambda_n + \frac{1}{3} - \frac{1}{6}(u(0)^2 + u(n)^2)$ that means

$$f_{opt} \leq \frac{2}{3} + \frac{1}{3}\cos\frac{\pi}{n+1} - \frac{1}{3}u(0)^2, \tag{13}$$

where we have used $u(0) = u(n)$ which should be hold from the symmetric property of $\tilde{A}$ and from that μ_n is the largest eigenvalue.

3 Large n analyses

Since the coefficients of Eq.(5) are not symmetric, the value θ shifts from $\frac{\pi}{n+1}$. We can denote the angle θ as $\theta = \frac{\pi}{n+1} + \delta(n)$, where $\delta(n)$ is expected to be $O(\frac{1}{n^2})$ in the limit of large n. Substituting $\theta = \frac{\pi}{n+1} + \delta(n)$ into Eq.(5), we find $\delta(n) = \frac{\pi}{3n^2}$ in the limit of large n. Therefore, in this limit we have up to $O(\frac{1}{n^3})$

$$f_{opt} = 1 - \frac{\pi^2}{6n^2} + \frac{2\pi^2}{9n^3}. \tag{14}$$

Our result satisfies $f_{opt} > 1 - \frac{2}{n^2}$. The right hand side of this inequality is the result by Franson et. al. [7], where the coefficients $c(i)$'s are not specified explicitly. In the limit of large n, $v(0)^2$ is estimated as

$$v(0)^2 = \frac{(\frac{\pi}{n})^2}{\frac{4n}{\pi}\int_0^\pi \sin^2 x dx} = \frac{\pi^2}{2n^3}. \tag{15}$$

In the same way we find

$$u(0)^2 = \frac{(\frac{\pi}{n})^2}{\frac{9n}{\pi}\int_0^\pi \sin^2 x dx} = \frac{2\pi^2}{9n^3}. \tag{16}$$

These results Eqs.(14)-(16) accord with the inequalities Eqs.(12), (13). We have $f_{opt} = f_{\mathbf{v}} + \frac{\pi^2}{18n^3}$ in the limit of large n.

We consider to teleport the state $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ that is the most difficult state to teleport[8]. Setting the coefficient vector $\mathbf{c}$ to be the optimal one $\mathbf{u}$, we have the following success probability

$$p_{\mathbf{u}}(|+\rangle) = \frac{1}{2}\Sigma_{i=0}^n (u(i) + u(i-1))^2 = {}^t\mathbf{u}A\mathbf{u} = {}^t\mathbf{u}(\frac{3}{2}\tilde{A} - \frac{1}{2}E + \frac{1}{4}\Gamma)\mathbf{u}. \tag{17}$$

In the limit of large n, $p_{\mathbf{u}}(|+\rangle)$ is estimated as

$$p_{\mathbf{u}}(|+\rangle) = \lambda_n - \frac{\pi^2}{12n^3}. \tag{18}$$

Therefore, we have, at least at large n, $p_{\mathbf{u}}(|+\rangle) < p_{\mathbf{v}}(|+\rangle) = \lambda_n$ as it should be, because, $\mathbf{v}$ is the vector that maximizes $p_{\mathbf{c}}(|+\rangle)$ [8].

4 State preparation

Here we show a simple way how to prepare the state $\Sigma_{j=0}^n c(j)|1\rangle^j|0\rangle^{n-j}|0\rangle^j|1\rangle^{n-j}$ starting from the original KLM state $\frac{1}{\sqrt{n+1}}\Sigma_{j=0}^n|1\rangle^j|0\rangle^{n-j}|0\rangle^j|1\rangle^{n-j}$ by using $2[\frac{n}{2}]$ beam splitters [9] as in Fig.1. The transmission coefficient t_i is settled as $t_i = \frac{c([\frac{n}{2}]-i)}{c([\frac{n}{2}]+1-i)}, i = 1, 2, \cdots, [\frac{n}{2}]$. If the $2[\frac{n}{2}]$ detectors detect no photons we obtain the state $\Sigma_{j=0}^n c(j)|1\rangle^j|0\rangle^{n-j}|0\rangle^j|1\rangle^{n-j}$ as output. The success probability is given by $\frac{1}{(n+1)(c([\frac{n}{2}]))^2}$, which is at least larger than $\frac{1}{n+1}$. When we use $\mathbf{v}$ as $\mathbf{c}$, $c([\frac{n}{2}])^2$ is estimated as $\frac{2}{n}$ in the limit of large n . This will mean that we can obtain the intended state with the probability around $\frac{1}{2}$.

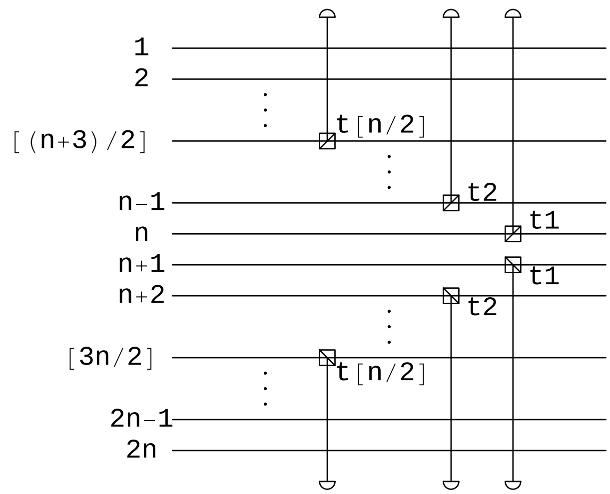

Fig. 1. A circuit of linear optics consisting of $2[\frac{n}{2}]$ beam splitters and corresponding photo detectors that probabilistically produces the intended state from the original KLM state.

5 Conclusion

We have examined the optimal fidelity $f_{opt} = \frac{2}{3} + \frac{1}{3}\cos\theta$, where θ is the smallest angle satisfying Eq.(5), for the high fidelity approach in the KLM scheme. We have identified the corresponding ancilla state. In the limit of large n we have found $f_{opt} = 1 - \frac{\pi^2}{6n^2} + \frac{2\pi^2}{9n^3}$ which slightly exceeds the result $f = 1 - \frac{2}{n^2}$ by Franson et. al.. We have examined another ancilla state that gives the maximal success probability for the state $|+\rangle$, which is the most difficult state to quantum teleport in the high fidelity approach. For this ancilla state we have the channel fidelity $f_{\mathbf{v}} = \frac{2}{3} + \frac{1}{3}\cos\frac{\pi}{n+1} - \frac{1}{3}v(0)^2$, where $v(0)$ is the first coefficient of the ancilla state. In the limit of large n we have $f_{opt} = f_{\mathbf{v}} + \frac{\pi^2}{18n^3}$. We also have exhibited an optical circuit producing a required ancilla state starting from the original KLM state.

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