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PRETTY GOOD STATE TRANSFER BETWEEN INTERNAL NODES OF PATHS

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In this paper, we show that, for any odd prime p and positive integer t, the path on $2^t p-1$ vertices admits pretty good state transfer between vertices a and $(n+1-a)$ for each a that is a multiple of 2^{t-1} with respect to the quantum walk model determined by the XY-Hamiltonian. This gives the first examples of pretty good state transfer occurring between internal vertices on an unweighted path, when it does not occur on the extremal vertices.

Keywords: quantum walks, state transfer, graph eigenvalues

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1 Introduction and Preliminary Definitions

Many quantum algorithms may be modelled as a quantum process occurring on a graph. In [1], Childs shows that any quantum computation can be encoded as a quantum walk in some graph and thus quantum walks can be regarded as a quantum computation primitive.

We model a network of n interacting qubits by a simple graph G on n vertices, where vertices correspond to qubits and edges to interactions. The interactions are defined by a time-independent Hamiltonian; that is, a symmetric matrix that acts on the Hilbert space of dimension 2^n. In this paper, we are concerned with the XY-Hamiltonian, whose action on the 1-excitation subspaces is equivalent to the action of the 01-symmetric adjacency matrix of G on $\mathbb{C}^n$. To be more specific, we consider the Hamiltonian

$$H = \sum_{ab \in E(G)} \sigma^x_{ab} \sigma^y_{ab}, \tag{1}$$

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where σ^x_{ab} and σ^y_{ab} are the operators that apply the Pauli matrices σ^x and σ^y at the qubits located at vertices a and b, and act as the identity at all other qubits. The sum is over all pairs of vertices a and b that are edges of the underlying graph. Consequently, if $|u\rangle$ stands for the system state in which the qubit at vertex u is at $|1\rangle$ and all others are at $|0\rangle$, then

$$H|a\rangle = \sum_{ab \in E(G)} |b\rangle. \tag{2}$$

Because of this, the action of H on the set $\{|a\rangle : a \in V(G)\}$ is equivalent to the action of the adjacency matrix $A(G)$ on the canonical basis of $\mathbb{C}^n$. In other words, H can be block diagonalized, and one of its blocks is equal to $A(G)$. The subspace spanned by $\{|a\rangle : a \in V(G)\}$ is the 1-excitation subspace of the Hilbert space, and the quantum dynamics restricted to this subspace corresponds to the scenario where one qubit, say the one at a, is initialized at $|1\rangle$ and all others at $|0\rangle$. Due to Schrdinger's equation

$$|\langle a| \exp(\mathrm{i}tH)|b\rangle|^2 \tag{3}$$

indicates the probability that the state $|1\rangle$ is measured at b after time t.

We are concerned solely with the 1-excitation subspace. From now on, $|u\rangle$ denotes the vector of the canonical basis of $\mathbb{C}^n$ that is 1 at the entry corresponding to vertex u at the ordering of the rows and columns of $A(G)$. From the remarks above, if the system is initialized with state $|1\rangle$ at vertex a and all others at $|0\rangle$, it follows that

$$|\langle a| \exp(\mathrm{i}tA)|b\rangle|^2 \tag{4}$$

indicates the probability that the state $|1\rangle$ is measured at b after time t. Whenever there is a time t such that this probability is equal to 1, we say that there is *perfect state transfer* between a and b. Perfect state transfer has been studied in many families of graphs, including circulants [2, 3], cubelike graphs [4, 5] and distance regular graphs [6]. See [7] for a survey of recent results.

For paths (or linear chains) with n vertices, perfect state transfer occurs if and only if n is 2 or 3. In general, perfect state transfer is understood to be a rare phenomenon, which motivates the definition of *pretty good state transfer* (or *almost state transfer*), which is said to occur if, for any $\epsilon > 0$, there is a time t such that

$$|\langle a| \exp(\mathrm{i}tA)|b\rangle| > 1 - \epsilon. \tag{5}$$

Pretty good state transfer was first studied for paths in [8] and [9]. Proofs and methods in this area are a mix of interesting applications of number theory and algebraic graph theory.

Pretty good state transfer between the end vertices in paths with n elements occurs if and only if $n+1$ is prime, twice a prime, or a power of 2 (see [8]). Moreover, in these cases, it occurs between any pair of vertices equally distant from the centre of the path. In contrast, the question of whether pretty good state transfer is possible between internal vertices in paths when it does not occur between the end vertices gives rise to a more interesting story and does not appear to have a simple answer. This question was raised in [10].

In this paper, we exhibit an infinite family of unweighted paths that admit pretty good state transfer between inner vertices but not between the two end vertices. These paths have

$2^t p - 1$ vertices where p is an odd prime and $t > 1$. We will make use of the following definitions and preliminary results.

Given a symmetric matrix M with d distinct eigenvalues $\theta_1 > ... > \theta_d$, we may write the spectral decomposition of M as follows:

$$M = \sum_{j=1}^{d} \theta_j E_j, \tag{6}$$

where E_r denotes the idempotent projection onto the eigenspace corresponding to θ_r. Note that $E_r E_s = \delta_{r,s} E_r$.

Given a vertex $a \in V(G) = \{1, ..., n\}$, let $|a\rangle$ denote the 01-vector that is 1 at the entry corresponding to a, and 0 elsewhere. We define the *eigenvalue support* of a as a subset of the distinct eigenvalues as follows:

$$\Theta_a = \{\theta_j : E_j|a\rangle \neq 0\}. \tag{7}$$

We say that vertices a and b are *strongly cospectral* if $E_j|a\rangle = \pm E_j|b\rangle$ for all idempotents E_j in the spectral decomposition. This property is necessary for both perfect and pretty good state transfer (see [10, Lemma 3]).

The spectrum of the adjacency matrix of the path (see [11] for example) on n vertices is

$$\theta_j = 2\cos\frac{\pi j}{n+1} \quad \text{for } j = 1, \ldots, n. \tag{8}$$

Note that they are indexed such that $\theta_1 > \ldots > \theta_n$. For each j, the eigenvector corresponding to θ_j is given by $(\beta_1, \ldots, \beta_n)$ where $\beta_k = \sin(k\pi j/(n+1))$. The following lemma immediately follows.

Lemma 1 *Vertices a and b of P_n are strongly cospectral if and only if $a + b = n + 1$.*

We will make use of the following result, which is derived from [10, Theorem 2], and completely characterizes pretty good state transfer. The second condition is an immediate consequence of Kronecker's theorem (see [10, Theorem 4]).

Theorem 1 *[10] Let a and b be vertices in a path with n vertices. Then pretty good state transfer happens between a and b if and only if both conditions below hold.*

(*i*) *$a+b = n+1$. In this case, for all $\theta_j \in \Theta_a$, define $\sigma_j = 0$ if $E_j|a\rangle = E_j|b\rangle$, and $\sigma_j = 1$ if $E_j|a\rangle = -E_j|b\rangle$.*

(*ii*) *For any set of integers $\{\ell_j : \theta_j \in \Theta_a\}$ such that*

$$\sum_{\theta_j \in \Theta_a} \ell_j \theta_j = 0 \quad \textit{and} \quad \sum_{\theta_j \in \Theta_a} \ell_j = 0, \tag{9}$$

then

$$\sum_{\theta_j \in \Theta_a} \ell_j \sigma_j \ \textit{is even.} \tag{10}$$

2 New Examples of Pretty Good State Transfer

The path graph P_n is the graph on vertices $\{1, \ldots, n\}$ where vertex a is adjacent to $a+1$ for all $a = 1, \ldots, n-1$. We will make use of some finite field theory in the proof of the next theorem.

A complex number ξ is a *primitive nth root of unity* if $\xi^n = 1$ and $\xi^k \neq 1$ for $k < n$. The *nth cyclotomic polynomial* is the unique monic polynomial $\Phi_n(x)$ with integer coefficients such that $\Phi_n(x)$ divides $x^n - 1$ and $\Phi_n(x)$ does not divide $x^k - 1$ for any $k < n$. The *nth cyclotomic field* is $\mathbb{Q}[x]/\Phi_n(x)$, the splitting field of the nth cyclotomic polynomial over the rationals. It is a standard fact that the nth cyclotomic field is isomorphic to the field extension of the rationals adjoining a primitive nth root of unity, $\mathbb{Q}(\zeta_n)$. See standard texts in number theory such as [12] for more details.

Theorem 2 *Given any odd prime p and positive integer t, there is pretty good state transfer in $P_{2^t p-1}$ between vertices a and $2^t p - a$, whenever a is a multiple of 2^{t-1}.*

Proof: For simplicity, let $n = 2^t p - 1$. For vertices a and $(n+1-a)$, condition (i) of Theorem 1 is satisfied with $2\sigma_j = 1 + (-1)^j$, by Lemma 1.

The eigenvalues of the path P_n belong to the cyclotomic field $\mathbb{Q}(\zeta_{2m})$, where $m = n+1$ and ζ_{2m} is a $2m$th primitive root of unity. More precisely, the eigenvalues of P_n are

$$2\cos\left(\frac{j\pi}{m}\right) = \zeta_{2m}^{j} + \zeta_{2m}^{-j} \in \mathbb{Q}(\zeta_{2m}). \tag{11}$$

If $m = 2^k p$, then the cyclotomic polynomial is

$$\Phi_{2m}(x) = \sum_{i=0}^{p-1} (-1)^i x^{2^k i}. \tag{12}$$

We will proceed by showing that part (ii) of Theorem 1 holds. If a is a multiple of 2^{t-1}, suppose there is a linear combination of the eigenvalues in Θ_a, satisfying

$$\sum_{j=1}^{n} \ell_j \theta_j = 0, \tag{13}$$

where we make $\ell_j = 0$ if $\theta_j \notin \Theta_a$. Recall that the ath entry of the θ_j-eigenvector is $\sin(a\pi j/(n+1))$. We see that θ_j belongs to Θ_a if and only if $2p$ does not divide j.

We define the polynomial $P(x)$ as follows:

$$P(x) = \sum_{j=1}^{n} \ell_j x^j + \sum_{j=n+2}^{2n+1} \ell_{2n+2-j} x^j \tag{14}$$

We see that ζ_{2m} is a root of $P(x)$ and, since $\Phi_{2m}(x)$ is the minimal polynomial of ζ_{2m}, we see that $\Phi_{2m}(x)$ divides $P(x)$.

Let $Q(x)$ be the following polynomial:

$$Q(x) = \sum_{j=1}^{2^t} \ell_j x^j + \sum_{j=2^t+1}^{2^t p-1} (\ell_j + \ell_{j-2^t}) x^j - \ell_{2^t(p-1)} x^{2^t p} + \sum_{j=1}^{2^t-1} (\ell_{2^t p-j} + \ell_{2^t(p-1)+j} - \ell_j) x^{2^t p+j}. \tag{15}$$

For a polynomial $p(x)$, we denote by $[x^t]p(x)$ the coefficient of x^t in $p(x)$. Consider $[x^k]\Phi_{2m}(x)Q(x)$. It is easy to see that $[x^k]\Phi_{2m}(x)Q(x) = [x^k]P(x)$ for $k = 0, \dots, 2^t(p+1)-1$. Since the degree of $Q(x)$ is $2^t(p+1)-1$, we may conclude that $Q(x)$ is the unique polynomial of degree $2^t(p+1)-1$ such that

$$[x^k]\Phi_{2m}(x)Q(x) = [x^k]P(x) \tag{16}$$

for $k = 0, \dots, 2^t(p+1)-1$. In particular, the quotient $P(x)/\Phi_{2m}(x)$ is a polynomial of degree $2^t(p+1)-1$ such that (16) holds, therefore

$$P(x) = \Phi_{2m}(x)Q(x). \tag{17}$$

From the coefficients of x^k for $k > 2^t(p+1)-1$, it follows that, for $j = 2, 4, \dots, 2^{t-1}-2$, and $i = 1, \dots, (p-1)/2$,

$$\ell_j - \ell_{2^t p-j} = (-1)^i(\ell_{i2^t \pm j} - \ell_{(p-i)2^t \mp j}), \text{ and} \tag{18}$$

$$\ell_{i2^{t-1}} - \ell_{(p-i)2^{t-1}} = 0. \tag{19}$$

Recall that $\ell_{2kp} = 0$ for any integer k.

Given $j \in \{2, 4, \dots, 2^{t-1}-2\}$, note that $j \not\equiv 0 \pmod p$, and since $2^t \not\equiv 0 \pmod p$, there is $i \in \mathbb{Z}_p$ such that $i2^t \equiv j \pmod p$. If $1 \le i \le (p-1)/2$, then $\ell_{i2^t-j} = \ell_{(p-i)2^t+j} = 0$, and if $(p-1)/2+1 \le i \le p-1$, then $\ell_{i2^t+j} = \ell_{(p-i)2^t-j} = 0$. In either case, it follows that $\ell_j - \ell_{2^t p-j} = 0$. Therefore $\ell_j = \ell_{2^t p-j}$ for all even j.

Thus, we see that

$$\sum_{\theta_j \in \Theta_a} \ell_j \sigma_j = \sum_{j \text{ even}} \ell_r \equiv \ell_{2p} \equiv 0 \pmod 2, \tag{20}$$

which concludes the proof $\square$.

3 Final Remarks

Pretty good state transfer between end vertices of paths was classified in [8]. In this paper, we have given an infinite family of paths where pretty good state transfer occurs between internal vertices, but not between the end vertices. When we first submitted this paper, we left as an open problem the full classification of the orders n where this occurs; this problem has since been solved by the third author in [13]. This gives a complete classification of pretty good state transfer in paths.

In [10], the authors study pretty good state transfer between the end vertices in a Heisenberg chain; they study the Heisenberg Hamiltonian, whose action on the 1-exication subspaces is equivalent to the action of the Laplacian adjacency matrix on $\mathbb{C}^n$. In this case, they show that pretty good state transfer occurs between the end vertices of P_n if and only if n is a power of 2. It is an open problem to find examples in which pretty good state transfer occurs between internal vertices according to the Heisenberg Hamiltonian when it does not occur between the end vertices.

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