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CRITERION FOR K-SEPARABILITY IN MIXED MULTIPARTITE STATES

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Using a recently introduced framework, we derive criteria for quantum k-separability, which are easily computed. In the case $k = 2$, our criteria are equally strong to the best methods known so far, while in all other cases there are currently no comparable criteria known. We also show how the criteria can be implemented experimentally.

Keywords: separability, entanglement detection, multipartite qudit system
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1 Introduction

Ever since quantum entanglement has started to be taken seriously by the scientific community, it has been subject to intensive studies and research. As soon as the properties of quantum mechanical systems were recognised not only to cause difficulties but also to allow for completely new applications and technology (which are far from comparison in classical physics, see e.g. Refs. [1, 2, 3, 4]), entanglement has played an important role in the rise of whole fields of research, such as quantum information theory.

Although the nature of bipartite quantum entanglement is being more and more understood [1], multipartite entanglement still holds many puzzling open questions. For example, although the concept of genuine multipartite entanglement is widely understood, tools for its detection are only beginning to be developed (see e.g. Refs. [5, 6, 7, 8]). Since genuine multipartite entanglement and partial separability are very important concepts for various quantum informational tasks (e.g. quantum secret sharing [1]) and especially the former is believed to also play an important role in nature (as for example in photosynthesis [9, 10]), constructing such tools is highly necessary. In particular, detection of genuine k-nonseparability in mixed states is essential, since it is necessary in order to distill genuinely k-nonseparable pure states [11].

This letter is organised as follows. In section 2, the basic terminology and definitions are

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reviewed, such that in section 3 we can present our criterion for k-separability (which is the main result of this letter). This criterion, along with several examples of its application, are then discussed in section 4 as is its potential experimental realisation in section 5.

2 Motivation and Definitions

In Ref. [7], a general framework was introduced, which allows for constructing very versatile separability criteria. In this letter, we will use this framework do derive necessary criteria for k-separability of quantum states. Firstly, the basic terminology needs to be defined.

Definition: An n-partite pure quantum state $|\Psi_{k-sep}\rangle$ is called k-separable, iff it can be written as a product of k substates:

$$|\Psi_{k-sep}\rangle = |\Psi_1\rangle \otimes |\Psi_2\rangle \otimes \cdots \otimes |\Psi_k\rangle \tag{1}$$

A mixed state ρ_{k-sep} is called k-separable, iff it has a decomposition into k-separable pure states:

$$\rho_{k-sep} = \sum_i p_i |\Psi^i_{k-sep}\rangle\langle\Psi^i_{k-sep}| \tag{2}$$

In particular, an n-partite state is called fully separable, iff it is n-separable. It is called genuinely n-partite entangled, iff it is not biseparable (2-separable). Note that the individual pure states composing a k-separable mixed state may be k-separable under different partitions. Hence, in general, k-separable mixed states are not separable w.r.t. any specific partition, which makes k-separability rather difficult to detect. Let us also remark that whenever a state is k-separable, it is automatically also k'-separable for all $k' < k$ (as illustrated in Fig. 1).

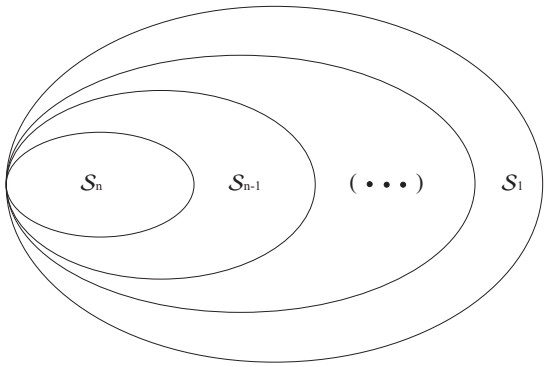

Fig. 1 Illustration of the geometry of the sets S_k, i.e. the sets of all k-separable states. Each set is convex and embedded within the next set: $S_n \subset S_{n-1} \subset \cdots \subset S_2 \subset S_1$.

In order to formulate our criterion for k-separability, we first need to define permutation operators P_i acting on two copies of an n-partite state. These operators swap the i-th subsystems of the two copies:

$$P_i|\Psi_{a_1,a_2,\cdots,a_n}\rangle \otimes |\Psi_{b_1,b_2,\cdots,b_n}\rangle = |\Psi_{a_1,a_2,\cdots,a_{i-1},b_i,a_{i+1},\cdots,a_n}\rangle \otimes |\Psi_{b_1,b_2,\cdots,b_{i-1},a_i,b_{i+1},\cdots,b_n}\rangle \tag{3}$$

where the a_j and b_j indicate the subsystems of the first and second copy of the state, respectively.
It is evident that if the permuted subsystem is separable from the rest of the state, the state is invariant under such a permutation. This observation is the key to constructing very general separability criteria.
In the following, permutation operators on sets of subsystems will be used, which is equivalent to applying several single permutation operators at once.

3 Criterion for k-separability

Now, we can state our main result:
Theorem: Every k-separable state ρ satisfies

$$\sqrt{\langle\Phi|\rho^{\otimes 2}P_{tot}|\Phi\rangle} - \sum_{\{\alpha\}}\left(\prod_{i=1}^{k}\langle\Phi|P_{\alpha_i}^{\dagger}\rho^{\otimes 2}P_{\alpha_i}|\Phi\rangle\right)^{\frac{1}{2k}} \leq 0 \qquad (*)$$

for all fully separable states $|\Phi\rangle$, where the sum runs over all possible partitions α of the considered system into k subsystems, the permutation operators P_{α_i} are the operators permuting the two copies of all subsystems contained in the i-th subset of the partition α and P_{tot} is the total permutation operator, permuting the two copies.

Proof: To prove this, observe that (like in Refs. [6, 7]) the left hand side of the inequality is a convex function of ρ (since the first term is the absolute value of a density matrix element and each term in the sum is the $2k$-th root of the product of $2k$ density matrix diagonal elements). Consequently, it suffices to prove the validity for pure states and validity for mixed states is guaranteed. So, let us assume w.l.o.g., that the given pure state ρ is k-separable w.r.t. the k-partition $\tilde{\alpha}$. Due to its separability, ρ is invariant under permutation of each element of $\tilde{\alpha}$:

$$P_{\tilde{\alpha}_i}^{\dagger}\rho^{\otimes 2}P_{\tilde{\alpha}_i} = \rho^{\otimes 2}. \qquad (4)$$

Therefore, the corresponding term in the sum can be written as

$$\left(\prod_{i=1}^{k}\langle\Phi|P_{\tilde{\alpha}_i}^{\dagger}\rho^{\otimes 2}P_{\tilde{\alpha}_i}|\Phi\rangle\right)^{\frac{1}{2k}} = \left(\prod_{i=1}^{k}\langle\Phi|\rho^{\otimes 2}|\Phi\rangle\right)^{\frac{1}{2k}} = \left(\prod_{i=1}^{k}\sqrt{\langle\phi_1|\rho|\phi_1\rangle\langle\phi_2|\rho|\phi_2\rangle}\right)^{\frac{1}{k}} = \sqrt{\langle\phi_1|\rho|\phi_1\rangle\langle\phi_2|\rho|\phi_2\rangle} \qquad (5)$$

where we use $|\Phi\rangle = |\phi_1\rangle \otimes |\phi_2\rangle$. Using $P_{tot}|\phi_1\rangle \otimes |\phi_2\rangle = |\phi_2\rangle \otimes |\phi_1\rangle$, we can now rewrite ineq. $(*)$ as

$$|\langle\phi_1|\rho|\phi_2\rangle| - \sqrt{\langle\phi_1|\rho|\phi_1\rangle\langle\phi_2|\rho|\phi_2\rangle} - \sum_{\{\alpha\neq\tilde{\alpha}\}}\left(\prod_{i=1}^{k}\langle\Phi|P_{\alpha_i}^{\dagger}\rho^{\otimes 2}P_{\alpha_i}|\Phi\rangle\right)^{\frac{1}{2k}} \leq 0 \qquad (6)$$

Now, the first term is an off-diagonal matrix-element and the second term is the squareroot of the product of the two corresponding diagonal elements, hence (and because ρ is a pure state), the first two terms cancel each other. It is evident that the remaining sum over strictly nonnegative terms (diagonal elements) with a negative sign is nonpositive, which proves our theorem. □

4 Results and Discussion

4.1 Example 1 - Three QuBits and Three QuTrits

Consider the family of three-qubit-states

$$\rho = \alpha|GHZ\rangle\langle GHZ| + \beta|W\rangle\langle W| + \frac{1-\alpha-\beta}{8}1 \tag{7}$$

with the well-known GHZ-state $|GHZ\rangle = (|000\rangle + |111\rangle)/\sqrt{2}$ and W-state $|W\rangle = (|001\rangle + |010\rangle + |100\rangle)/\sqrt{3}$.
It turns out that for states in the vicinity of the GHZ-state, $|\Phi\rangle = |000111\rangle$ is a good choice, while the W-state is not detected at all in the computational basis. On the other hand, the W state can be detected in a basis rotated by $\pi/4$, where in turn the GHZ-state is not detected. In Fig. 2(a) the detection quality of ineq. $(*)$ is illustrated for both these choices for $|\Phi\rangle$.

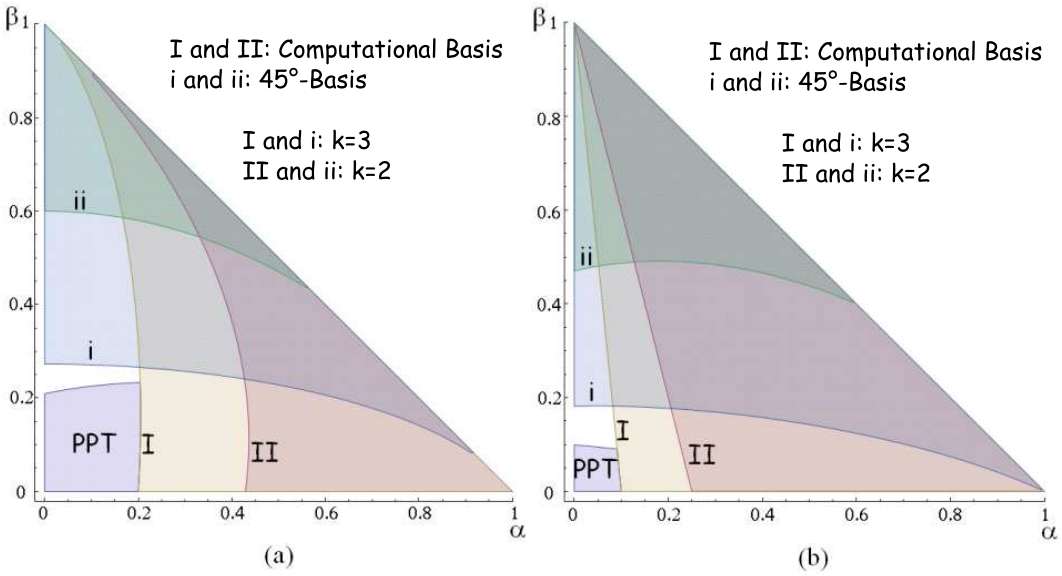

Fig. 2 (Colour online): Illustration of the detection parameters of (a) the three-qubit-state (7) and (b) the three-qutrit-state (8) for the inequality $(*)$. The lines I and II represent the thresholds of detection for entanglement ($k = 3$) and genuine tripartite entanglement ($k = 2$) in the computational basis. The lines i and ii represent the same in the basis rotated by (a) $\pi/4$ and (b) $\pi/4$ around two axes.

Consider now the following generalisation of these states to three-qutrit-systems:

$$\rho = \alpha|gGHZ\rangle\langle gGHZ| + \beta|\xi\rangle\langle\xi| + \frac{1-\alpha-\beta}{27}1 \tag{8}$$

with the generalised GHZ-state $|gGHZ\rangle = (|000\rangle + |111\rangle + |222\rangle)/\sqrt{3}$ and the state $|\xi\rangle = (|012\rangle + |021\rangle + |102\rangle + |120\rangle + |201\rangle + |210\rangle)/\sqrt{6}$, which is a possible generalisation of the W-state (as it also is genuinely 3-partite entangled while having entangled reduced density

matrices). Concerning the choice of $|\Phi\rangle$, the same arguments as above apply and Fig. 2(b) shows the detection quality of our criterion in this case (where the rotated basis is rotated around the y- and z-axis by $\pi/4$ each).

4.2 Example 2 - General GHZ-States

The most general maximally entangled state for n qudits has the form

$$|\Psi\rangle = \sum_{i=0}^{d-1} |i\rangle^{\otimes n}. \tag{9}$$

Consider this state dampened by isotropic noise, i.e. the family

$$\rho = p|\Psi\rangle\langle\Psi| + \frac{1-p}{d^n} 1_{d^n} \tag{10}$$

Straightforward computation of the criterion $(*)$ yields that these states are not k-separable if

$$p > \frac{\gamma}{\gamma + d^{n-1}} \tag{11}$$

where γ is the number of possible partitions, i.e. depends only on k and n.
Note that for general entanglement detection ($k = n$), this criterion is as strong as the PPT criterion, since $\gamma = 1$ and thus ρ is detected to be entangled for $p > (1 + d^{n-1})^{-1}$, which is exactly the threshold detected by the PPT criterion.

4.3 Example 3 - Spin Chain

A multipartite quantum system cannot exhibit arbitrary entanglement properties. This has several implications in the properties of spin chains and spin lattices, which are the typical subjects of statistical and solid state physics (see e.g. the review article [12]). As an example, let us consider translationally invariant states of an infinite one dimensional chain of qubits [13, 14, 15], also known as finitely correlated states, which are ground states of Hamiltonians, and intensively studied in literature.
In Ref [15], the authors investigated the maximally possible achievable entanglement of nearest neighbours, which serves as a reference point for interpreting entanglement values obtained for real physical systems and sheds light onto the subtle properties of spin-spin interactions. An n-qubit state of an infinite translationally invariant spin-$\frac{1}{2}$ chain is given by

$$\rho_{[1,2,\ldots,n]} = \sum_{\mathbf{s},\mathbf{t}} |\mathbf{s}\rangle\langle\mathbf{t}| \mathrm{Tr}(v_{\mathbf{s}}^\dagger \, \rho_B \, v_{\mathbf{t}}) \tag{12}$$

with $|\mathbf{s}\rangle = |s_1 \ldots s_n\rangle$, $v_{\mathbf{s}} = v_{s_1} \cdots v_{s_n}$ and $s_i = 0, 1$. Here the operators v_0, v_1 and the auxiliary state ρ_B are of dimension b and have to satisfy certain equations. They simulate the "rest of the chain" and, consequently, the dimensionality b approximates the qubit, two–qubit, ..., n–qubit reduced states of the pure state of the infinite qubit chain accordantly (i.e. the higher b, the better the approximation).
We want to find out whether the three–, four– or five–qubit states of this infinite chain with maximal nearest neighbour entanglement are k-separable or for which k our inequality is

violated. For that, we use the values given in Ref. [15] for the case of maximised nearest neighbour entanglement for dimension $b = 2$ and $b = 6$ (which was shown to be a good approximation). We find that for all $k \leq n$ the inequalities $(*)$ are violated, and thus the chain is not k-separable for any k, but genuinely multipartite entangled.

4.4 Discussion

It is quite clear from the above examples, that the introduced criteria are not only the first criteria for genuine k-nonseparability, but also work very well as criteria for detecting general quantum entanglement.
Although a k-separable state is always also k'-separable for all $k' < k$, our criteria do not include one another, i.e. if a state violates inequality $(*)$ for a certain k_1, there may still be a certain $k_2 > k_1$ for which the inequality is satisfied, even though the state is evidently not k_2-separable. It is unclear wether this can be exploited to improve the criterion (e.g. by using that the m-fold tensor product of a k-separable state is $(k \times m)$-separable).

All our achieved results can be improved by optimising the used state $|\Phi\rangle$ over all local unitary transformations, i.e. transformations of the form $\rho \rightarrow U\rho U^\dagger$, where $U = U_1 \otimes U_2 \otimes \cdots \otimes U_n$. Although such an optimisation is a highly complex task, it can be done comperatively easily by means of the methods described in Refs. [16, 17]. In many cases however, this is unnecessary, since our criteria are quite strong even without optimisation.

Since at present there are no comparable general criteria for detecting k-nonseparability, it is hard to estimate how tight the criterion presented in this paper works. The only other criterion known to the authors is applicable only to systems of multiple qubits and detects significantly less k-nonseparability [18, 19].
While our criterion most certainly does not detect *all* k-inseparable states as such, the actual volume of these undetected states is yet unknown.

5 Experimental implementation

Since a full quantum state tomography requires a huge number of measurements, it is important for separability criteria of multipartite systems to be experimentally implementable without such a procedure. With inequality $(*)$ this is possible, due to the fact that it can be expressed in terms of density matrix elements, each of which can be measured efficiently with a single observable. For any fixed $|\Phi\rangle$, the number of density matrix elements in ineq. $(*)$ is $2^n - 1$, which – compared to the $d^{2n}/2$ measurements needed for a full tomography – not only grows significantly slower with n, but more importantly has the vast advantage of being independent of the dimensionality d of the subsystems.
The observables follow directly from the choice of $|\Phi\rangle$. For the observables associated with the second term in ineq. $(*)$, one only needs to measure diagonal density matrix elements

$$\mathcal{O}_{\alpha_i}^{A/B} = |\chi_{\alpha_i}^{A/B}\rangle\langle\chi_{\alpha_i}^{A/B}|, \tag{13}$$

where

$$|\chi_{\alpha_i}^{A}\rangle \otimes |\chi_{\alpha_i}^{B}\rangle = P_{\alpha_i}|\Phi\rangle \tag{14}$$

and therefore

$$\langle \mathcal{O}^A_{\alpha_i} \otimes \mathcal{O}^B_{\alpha_i} \rangle = \langle \Phi | P^\dagger_{\alpha_i} \rho^{\otimes 2} P_{\alpha_i} | \Phi \rangle. \tag{15}$$

From the separability of $|\Phi\rangle$ follows the product form of the observables

$$\mathcal{O}^{A/B}_{\alpha_i} = \bigotimes_{j=1}^{n} O_j. \tag{16}$$

These observables can be implemented by means of local observables O_j of the subsystems. For the positive term of the inequality, one needs to measure a single off-diagonal density matrix element, which is also achievable via a single multipartite coherence measurement. An efficient way to do so is to measure the n-party visibility (see Refs. [20, 21, 22]), which can be done using only local observables.

6 Conclusion

We have developed an easily computable criterion for detecting k-nonseparability in mixed states for arbitrary quantum systems and arbitrary values of k, which does not involve eigenvalue computation and works well even without numerical optimisation. In cases where comparable results exist (i.e. for the special case of detecting genuine multipartite entanglement, $k = 2$) it is as good as the best known methods so far (i.e. the methods presented in [6] and [7]). Also, for the case $k = n$, i.e. general detection of entanglement, in many cases it is as strong as the PPT-criterion. We also discussed possibilities to implement our criterion experimentally.

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