Quantum Information and Computation, Vol. 7, No. 7 (2007) 584–593
© Rinton Press

GENUINE TRIPARTITE ENTANGLEMENT SEMI-MONOTONE FOR $(2 \times 2 \times n)$-DIMENTIONAL SYSTEMS

CHANG-SHUI YU[1], HE-SHAN SONG[1], and YA-HONG WANG[1,2]
[1] *Department of Physics, Dalian University of Technology, Dalian 116024, P.R. China*
[2] *School of Information Science and Engineering, Dalian Inst of Light Industry, Dalian 116034, P.R. China*

Received August 31, 2006
Revised January 16, 2007

In this paper, we present a new approach to study genuine tripartite entanglement existing in $(2 \times 2 \times n)-$dimensional quantum pure states. By utilizing the approach, we introduce a particular quantity to measure genuine tripartite entanglement. The quantity is shown to be an entanglement monotone in 2-dimensional subsystems (semi-monotone) and reaches zero for separable states and $(2 \times 2 \times 2)-$dimensional W states, hence is a good criterion to characterize genuine tripartite entanglement. Furthermore, the formulation for pure states can be conveniently extended to the case of mixed states by utilizing the kronecker product approximation technique. As applications, we give the analytic approximation for weakly mixed states, and study the genuine tripartite entanglement of two given weakly mixed states.

Keywords: Entanglement, tripartite entanglement, entanglement measure
Communicated by: S Braunstein & C Fuchs

1 Introduction

Entanglement is a valuable physical resource for many quantum information processing, such as quantum computation [1], quantum cryptography [2], quantum teleportation [3], quantum dense coding [4] and so on. The understanding of entanglement is at the very heart of quantum information theory. Recently, many efforts have been made to characterize quantitatively the entanglement properties of a quantum system[5-8], however, the good understanding is only restricted in low-dimensional systems. The quantification of entanglement for higher dimensional systems and multipartite quantum systems remains to be an open question.

Since the remarkable concurrence was presented [5], it has been shown to be a useful entanglement measure for the systems of qubits. Based on the concurrence, V. Coffman et al [9] introduced the so called residual entanglement for tripartite systems of qubits and shew that the residual entanglement can be employed to measure genuine tripartite entanglement, which opens the path to studying multipartite entanglement. However, unlike the entanglement in low-dimensional systems, the entanglement in high-dimensional or multipartite systems is much more complicated. E.g. Dür et al [10] have shown that three qubits can be entangled in two inequivalent ways; Miyake [11,12] has shown that the multipartite entanglement can be divided into more classes based on the hyperdeterminant. These inequivalent entanglement classes tell us that a single quantity can not effectively and thoroughly measure entanglement of a high-dimensional or multipartite systems. However, for some particular purpose, one can still characterize entanglement by only a single quantity. For example: The most naturally,

one can measure entanglement of a certain class by only a single quantity [11-14]; One can employ only a quantity to study the separable property of a given quantum system [15,16]; One can also collect the contributions of some different entanglements as a whole to study the correlations between subsystems [17-20]; And so on.

In this paper, again a single quantity denoted by τ is presented in terms of a new approach to characterize the genuine tripartite entanglement for $(2\times 2\times n)-$dimensional quantum systems. The distinct advantage of τ is that it can not only characterize the properties of genuine tripartite entanglement existing in a given quantum pure state and be conveniently extended to mixed states by kronecker product approximation technique, but it is a entanglement semi-monotone, i.e. it is an entanglement monotone considering the two 2-dimensional subsystems and invariant under local unitary transformations in the higher-dimensional subsystem. In this sense, if the usual Positive Operation-Valued Measures (POVM's) on the higher-dimensional subsystem is not considered, τ is even a good entanglement measure. Furthermore, one will also find that the initial residual entanglement introduced in Ref. [9] can be obtained by our approach. In this sense, we also consider that τ is a generalization of the initial residual entanglement. As applications, we give the analytic approximation of τ for weakly mixed tripartite quantum states (quasi pure states) and consider the genuine tripartite entanglement of some quasi pure states, which shows the sufficiency of our measure as a criterion to test entanglement and the workability as an indicator of entanglement in these cases. Note that even though there are other results [12,21,22] for $(2\times 2\times n)-$dimensional quantum systems, they are essentially different from ours. For example, Ref. [21-22] studied the entanglement of assistance which is some kind of bipartite entanglement in fact. Ref. [12] mainly focused on the classification of multipartite entanglement and so far it seemed difficult to obtain an operational entanglement evaluation for mixed states. The paper is organized as follows. First, we give τ for pure states by a new approach and prove that τ is a entanglement semi-monotone and can characterize genuine tripartite entanglement; and then we extend it to mixed states and discuss the genuine tripartite entanglement of some quasi pure states; the conclusions are drawn in the end.

2 The genuine tripartite entanglement semi-monotone for pure states

At first, let us introduce the concept of "tilde inner products". The concept was presented by Wootters to introduce the remarkable concurrence in Ref. [5]. Considering any two bipartite state vectors of qubits $|x\rangle$ and $|y\rangle$, the tilde inner product of $|x\rangle$ and $|y\rangle$ is defined by

$$\langle x|\,\tilde{y}\rangle = \langle x|\,\sigma_y\otimes\sigma_y\,|y^*\rangle\,, \tag{1}$$

where $|\tilde{y}\rangle = \sigma_y\otimes\sigma_y\,|y^*\rangle$ with $|y^*\rangle$ is the complex conjugate of $|y\rangle$ and $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$. However, for convenience, whenever the tilde inner product is mentioned, we refer to

$$(\langle x|\,\tilde{y}\rangle)^* = \langle x^*|\,\sigma_y\otimes\sigma_y\,|y\rangle\,. \tag{2}$$

Now, let us focus on $(2\times 2\times n)$ $-$dimensional tripartite quantum pure state $|\psi_{ABC}\rangle$ defined in the Hilbert space $H_1\times H_2\times H_3$, which can be written in the standard basis by

$$|\psi_{ABC}\rangle = \sum_{i,j=0}^{1}\sum_{k=0}^{n-1} a_{ijk}\,|i\rangle_A\,|j\rangle_B\,|k\rangle_C = \sum_{k=0}^{n-1}|\varphi_k\rangle\,|k\rangle\,, \tag{3}$$

where $|\varphi_k\rangle = \sum_{i,j=0}^{1} a_{ijk} |i\rangle_A |j\rangle_B$ corresponds to $|k\rangle$ $(= |k\rangle_C)$ of the party C. For any a group of basis $\{|\phi_l\rangle\}, l = 0, 1, \cdots, n-1$, defined in H_3, one can always project $|\psi_{ABC}\rangle$ onto them, and obtain correspondingly an unnormalized bipartite pure state of qubits defined in $H_1 \times H_2$. Without loss of generality, here we choose $\{|\phi_l\rangle\} = \{|k\rangle\}$, therefore one can obtain a corresponding set of unnormalized bipartite pure states $\{|\varphi_k\rangle\}$. Arranging these bipartite states in terms of the order of $|k\rangle$, we can construct a matrix Φ given by

$$\Phi = [|\varphi_0\rangle, |\varphi_1\rangle, \cdots, |\varphi_{n-1}\rangle], \tag{4}$$

where it is implied that $|\varphi_k\rangle$ have been considered as column vectors. Hence, on the basis of the tilde inner product, we can obtain a new matrix $\mathcal{M}$ given by

$$\mathcal{M} = \Phi^T \sigma_y \otimes \sigma_y \Phi, \tag{5}$$

where each element $\mathcal{M}_{ij}$ denotes the tilde inner product of two bipartite pure states $|\varphi_i\rangle$ and $|\varphi_j\rangle$. Furthermore, one will find that $\mathcal{M}$ includes important information: First, because each column of Φ is not normalized, Φ includes the information of probabilities with which one can obtain the column vectors from $|\psi_{ABC}\rangle$; Second, due to the tilde inner product, $\mathcal{M}$, in particular its diagonal elements, includes the separability information of the columns of Φ. Since $\mathcal{M}$ is a matrix defined in $(n \times n)-$dimensional Hilbert space, we can always consider $\mathcal{M}$ as a $(n \times n)-$dimensional bipartite quantum pure state of qudits, which is given in matrix form. Therefore, it is natural to consider the entanglement measure of such an abstract bipartite pure state [a].

As we know, I-concurrence [23,8] $C_I(|\Psi\rangle)$ is a good measure for bipartite quantum pure state $|\Psi\rangle$ defined in arbitrary dimension, which has been shown to be the length of the concurrence vector [7] by W. K. Wootters [24]. We can describe them by an equation as

$$C_I(|\Psi\rangle) = \sqrt{2[|\langle\Psi|\,\Psi\rangle|^2 - tr(\rho_r^2)]} = \sqrt{\sum_{\alpha,\beta=1}^{n(n-1)/2} |C_{\alpha\beta}|^2}, \tag{6}$$

where ρ_r denotes the reduced density matrix by tracing over one of the two systems, $C_{\alpha\beta} = \langle\Psi^*|\, s_\alpha \otimes s_\beta\, |\Psi\rangle$ with s_α and s_β the generators of $SO(n)$. Applying the measure to the state $\mathcal{M}$, we can obtain

$$C_I(\mathcal{M}) = \sqrt{2\left\{[tr\,(\mathcal{M}\mathcal{M}^\dagger)]^2 - tr\left[(\mathcal{M}\mathcal{M}^\dagger)^2\right]\right\}}, \tag{7}$$

where tr denotes trace operation.

As a consequence, we can summarize all above and obtain a following formal expression.

Theorem 1. The genuine tripartite entanglement semi- monotone τ for the given pure state $|\psi_{ABC}\rangle$ can be obtained by

$$\tau = \tau\,(|\psi_{ABC}\rangle) = [C_I(\mathcal{M})]^{\frac{1}{2}}, \tag{8}$$

[a] If $\mathcal{M}$ is considered as an unnormalized pure state separately, it will make no sense to measure the entanglement because there will exist a undetermined constant (the normalization constant). However, $\mathcal{M}$ is not separated here, but closely related to the normalized $|\psi_{ABC}\rangle$. That is to say, although the constant is not determined for $\mathcal{M}$, it is determined for $|\psi_{ABC}\rangle$ which is what we care for. Furthermore, the normalization constant of $\mathcal{M}$ also includes valuable information mentioned in the text and has its real value. Therefore, $\mathcal{M}$ should not be normalized.

where the corresponding parameters have been given above.

What's more, consider the expression of the second "=" in eq. (6), we can expand τ by

$$\begin{aligned}\tau &= \Bigg[\sum_{i\neq l, j\neq k} |\langle\psi^*_{ABC}|\,\sigma_y\otimes\sigma_y\otimes e_{ij}\,|\psi_{ABC}\rangle \times \langle\psi^*_{ABC}|\,\sigma_y\otimes\sigma_y\otimes e_{lk}\,|\psi_{ABC}\rangle \\ &\quad - \langle\psi^*_{ABC}|\,\sigma_y\otimes\sigma_y\otimes e_{ik}\,|\psi_{ABC}\rangle \times \langle\psi^*_{ABC}|\,\sigma_y\otimes\sigma_y\otimes e_{lj}\,|\psi_{ABC}\rangle|^2\Bigg]^{\frac{1}{4}}, \qquad (9)\end{aligned}$$

where $e_{ij} = |i\rangle\langle j|$ with $|i\rangle$ denoting the standard basis of party C.

Proof. First of all, one can easily justify the following

Remark: τ given in above procedure can be reduced to the $(2\times 2\times 2)$-dimensional case given in Ref. [9]. Noting that a constant difference is neglectable.

Next, we will prove the theorem by two steps. At first, we will prove that τ is an entanglement semi-monotone, and then we will show that τ can characterize genuine tripartite entanglement for $(2\times 2\times n)$ −dimensional pure state $|\psi_{ABC}\rangle$.

Entanglement semi-monotone. We first note that τ is invariant under permutations of the two parties A and B of $|\psi_{ABC}\rangle$ defined in 2-dimensional Hilbert space respectively, hence we employ the method given in Ref. [10] to prove that τ is non-increasing under local operations assisted with classical communication (LOCC) in party A only. Due to the same reason mentioned in Ref. [10], we also consider a sequence of two-outcome POVM's. Let A_1 and A_2 be the two POVM elements such that $\mathrm{A}_1^\dagger A_1 + A_2^\dagger A_2 = \mathbf{1}_2$, with $\mathbf{1}_\delta$ denoting δ-dimensional identity matrix, then $A_i = U_i D_i V$, where U_i and V are unitary matrices and D_i are diagonal matrices with entries (a, b) and $[\sqrt{1-a^2}, \sqrt{1-b^2}]$, respectively. For some tripartite initial state $|\Psi\rangle$, let $|\Theta_i\rangle = (A_i\otimes\mathbf{1}_2\otimes\mathbf{1}_n)\,|\Psi\rangle$ be the unnormalized states obtained after the POVM operations. The corresponding normalized states can be given by $|\Psi'_i\rangle = |\Theta_i\rangle/\sqrt{p_i}$, where $p_i = \langle\Theta_i|\,\Theta_i\rangle$. Then

$$\langle\tau\rangle = p_1\tau(|\Psi'_1\rangle) + p_2\tau(|\Psi'_2\rangle). \qquad (10)$$

Considering the expression of a tripartite quantum state given in eq. (3), $|\Psi'_i\rangle$ can also be rewritten by

$$|\Psi'_i\rangle = \sum_{k=0}^{n-1}\left(\frac{(A_i\otimes\mathbf{1}_2)\,|\varphi_k\rangle}{\sqrt{p_i}}\right)|k\rangle\,. \qquad (11)$$

Hence, after operation A_1, $\mathcal{M}'$ corresponding to the tilde inner products can be constructed by

$$\begin{aligned}\mathcal{M}'_{jk} &= \frac{1}{p_1}\langle\varphi^*_j|\left[\left(V^T D_1 U_1^T\right)\otimes\mathbf{1_2}\right]\sigma_y\otimes\sigma_y\left[(U_1 D_1 V)\otimes\mathbf{1_2}\right]|\varphi_k\rangle \\ &= \pm\frac{ab}{p_1}\langle\varphi^*_j|\,\sigma_y\otimes\sigma_y\,|\varphi_k\rangle = \pm\frac{ab}{p_1}\mathcal{M}_{jk}. \qquad (12)\end{aligned}$$

Namely,

$$\mathcal{M}' = \pm\frac{ab}{p_1}\mathcal{M}. \qquad (13)$$

Analogously, $\mathcal{M}''$ corresponding to A_2 can be given by

$$\mathcal{M}'' = \pm \frac{\sqrt{(1-a^2)(1-b^2)}}{p_2} \mathcal{M}. \tag{14}$$

On the basis of eq. (7) and eq. (8), one can obtain

$$\tau(|\Psi_1'\rangle) = \frac{ab}{p_1} \tau(|\Psi\rangle), \tau(|\Psi_2'\rangle) = \frac{\sqrt{(1-a^2)(1-b^2)}}{p_2} \tau(|\Psi\rangle). \tag{15}$$

Substituting eq. (15) into eq. (10), according to Ref. [10], one can obtain that $\langle \tau \rangle \leq \tau(|\Psi\rangle)$. What's more, eq. (12) implies that there may be an overall phase difference for $\mathcal{M}$ if a local unitary transformation on party A is considered. That is to say, $\tau(|\Psi\rangle)$ will be invariant under such local unitary transformations.

Now, let us focus on the third party C. Any a given $n \times n$ matrix Q operated on party C of $|\psi_{ABC}\rangle$ can be described by $(\mathbf{1}_2 \otimes \mathbf{1}_2 \otimes Q)\,|\psi_{ABC}\rangle$ denoted by $|\psi'_{ABC}\rangle$. Based on the tilde inner products, one can always construct the corresponding matrix $\tilde{\mathcal{M}}$ following above procedure. Consider the standard basis $\{|k\rangle\}$ of party C in H_3, $\tilde{\mathcal{M}}$ can be written by

$$\tilde{\mathcal{M}}_{ij} = \sum_{l,m=0}^{n-1} \langle \varphi_l^* |\, \sigma_y \otimes \sigma_y \,|\varphi_m\rangle\, Q_{il} Q_{jm}, \tag{16}$$

where $|\varphi_j\rangle$ are defined the same to those in eq. (3). If we operate Q on $\mathcal{M}$ by $Q^T \mathcal{M} Q$, considering the same basis $\{|k\rangle\}$, one can obtain

$$\left[Q^T \mathcal{M} Q\right]_{lm} = \sum_{i,j=0}^{n-1} \langle \varphi_i^* |\, \sigma_y \otimes \sigma_y \,|\varphi_j\rangle\, Q_{il} Q_{jm}, \tag{17}$$

where $[\cdot]_{lm}$ denote the entries of the corresponding matrix. Hence, from eq. (16) and eq. (17), one can get

$$\tilde{\mathcal{M}} = Q^T \mathcal{M}^T Q. \tag{18}$$

Since $\mathcal{M}$ can be regarded as a bipartite pure state in matrix form, we can assume that $\mathcal{M}$ is defined in the Hilbert space $H_3 \otimes H_3'$ and denotes an entangled state of parties C and C'. According to eq. (18), we can draw a conclusion that for the matrix $\mathcal{M}$ of the tilde inner products, operating a transformation Q on party C of $|\psi_{ABC}\rangle$ is equivalent to operating $Q^T \otimes Q^T$ on the abstract state $\mathcal{M}^T$ which is defined in $H_3' \otimes H_3$. In other words, considering the local operations on party C of $|\psi_{ABC}\rangle$ is equivalent to considering the local operations on $\mathcal{M}^T$. If Q is a unitary transformation, one can easily find that the entanglement of $\mathcal{M}$ measured by eq. (7) is invariant, i.e. τ is invariant under local unitary transformation. If Q is a usual POVM, one will not ensure that τ is not always increasing. That is to say, τ is an entanglement semi-monotone. This completes the first step.

Characterizing genuine tripartite entanglement. Let us first show that $\tau = 0$ for semiseparable pure states and low-local-rank W states [11]. Considering the invariant permutation of A and B, any semiseparable pure state $|\psi_{ABC}\rangle$ can be given by

$$|\psi_{ABC}\rangle = |\varphi_{AB}\rangle \otimes |\chi_C\rangle \tag{19}$$

or

$$|\psi'_{ABC}\rangle = |\chi_A\rangle \otimes |\varphi_{BC}\rangle\,, \tag{20}$$

where $|\chi_i\rangle$ denote the quantum pure states for the ith single party and $|\varphi_{pq}\rangle$ denote the bipartite pure states for the pth and the qth parties. If projecting the state $|\psi_{ABC}\rangle$ given by eq. (19) onto any a group of basis of H_3 corresponding to party C, one can obtain that the corresponding matrix $\mathcal{M}$ of the tilde inner products has the entries given by

$$\mathcal{M}_{ij} = \langle i\,|\chi_C\rangle\,\langle j\,|\chi_C\rangle\,\langle\varphi^*_{AB}|\,\sigma_y \otimes \sigma_y\,|\varphi_{AB}\rangle\,. \tag{21}$$

According to eq. (7) and eq. (8), it is obvious that $\tau = 0$ in this case. If projecting the state $|\psi'_{ABC}\rangle$ given by eq. (20) onto any a group of basis $\{|\phi_l\rangle\}$ of H_3, one can obtain the corresponding matrix

$$\mathcal{M}'_{ij} = (\langle\chi^*_A| \otimes \langle\kappa^*_{Bi}|)\,\sigma_y \otimes \sigma_y\,(|\chi_A\rangle \otimes |\kappa_{Bj}\rangle) = 0, \tag{22}$$

where $|\kappa_{Bj}\rangle = (\mathbf{1}_2 \otimes \langle\phi_j|)\,|\varphi_{BC}\rangle$. Therefore, one can easily obtain that $\tau = 0$ for $|\psi'_{ABC}\rangle$. The W states, the local rank of which being $(2,2,2)$ is required, can always be reduced to a $(2\times2\times2)$-dimensional subspace of $H_1 \otimes H_2 \otimes H_3$ and be considered as tripartite pure states of qubits. According to the *Remark* made in the proof of theorem 1, one can have $\tau = 0$ for such W states. It should be noted that for the W states with high local rank, we believe they own genuine tripartite entanglement [25]. It is reasonable. As we know, Ref. [11] has introduced the onionlike classification of multipartite quantum states. The classification shows that the quantum states in the outer class can always be converted irreversibly into those in the inner class. Hence, we can say the outer classes "include" the inner ones. GHZ class with local rank $(2,2,2)$ as the innermost class to characterize genuine tripartite entanglement is hence "included" by outer class. In this sense, we can safely say that $\tau \neq 0$ for the W states with high local rank. It is also in this sense that τ can be believed to be the generalization of the initial residual entanglement.

Next, we will show that $\tau \neq 0$ for any quantum state with genuine tripartite entanglement. According to the tensor treatment of $|\psi_{ABC}\rangle$ [15], $|\psi_{ABC}\rangle$ can be regarded as a tensor grid, whose units can be considered to be tensor cubic. If there exist genuine tripartite entanglements in $|\psi_{ABC}\rangle$, there must exist at least such a tensor cubic of the grid as has genuine tripartite entanglement. In other words, based on eq. (6), there must exist some integers α^* and β^*, such that $|C_{\alpha^*\beta^*}|^2 \neq 0$. Hence, we can draw the conclusion that $\tau = 0$ means that there does not exist any genuine tripartite entanglement in $|\psi_{ABC}\rangle$. This completes the second step.□

3 Extension to mixed states

Consider $\tau\,(|\psi_{ABC}\rangle)$ of pure states, the corresponding quantity of mixed states ρ is then given as the convex roof

$$\tau(\rho) = \inf \sum_i p_i \tau(|\gamma_i\rangle) \tag{23}$$

of all possible decompositions into pure states $|\gamma_i\rangle$ with

$$\rho = \sum_i p_i\,|\gamma_i\rangle\,\langle\gamma_i|\,, p_i \geq 0. \tag{24}$$

$\tau(\rho)$ vanishes if and only if ρ does not include any genuine tripartite entanglement. According to the matrix notation [7] of equation (24), one can obtain $\rho = \Gamma W \Gamma^\dagger$, where W is a diagonal matrix with $W_{ii} = p_i$, the columns of the matrix Γ correspond to the vectors $|\gamma_i\rangle$. Due to the eigenvalue decomposition: $\rho = \Phi M \Phi^\dagger$, where M is a diagonal matrix whose diagonal elements are the eigenvalues of ρ, and Φ is a unitary matrix whose columns are the eigenvectors of ρ, one can obtain $\Gamma W^{1/2} = \Phi M^{1/2} U$, where $U \in C^{r\times N}$ is a Right-unitary matrix, with N and r being the column number of Γ and the rank of ρ. Therefore, based on the matrix notation and eq. (9), eq. (23) can be directly rewritten in a twice-doubled Hilbert space as

$$\tau(\rho) = \inf_U \sum_{i=1}^{N} \left\{ \left[\left(U^T \otimes U^\dagger \otimes U^T \otimes U^\dagger \right) \times \mathcal{A} \left(U \otimes U^* \otimes U \otimes U^* \right) \right]_{ii,ii}^{ii,ii} \right\}^{\frac{1}{4}}, \tag{25}$$

where

$$\mathcal{A} = \left(\varrho^{1/2} \right)^T \sum_{i\neq l, j\neq k} A_{ijkl} \left(\varrho^{1/2} \right) \tag{26}$$

with

$$\varrho^{1/2} = \left(\Phi M^{1/2} \right)^T \otimes \left(\Phi M^{1/2} \right)^\dagger \otimes \left(\Phi M^{1/2} \right)^T \otimes \left(\Phi M^{1/2} \right)^\dagger, \tag{27}$$

$$\begin{aligned} A_{ijkl} = {} & \Sigma_{ij} \otimes \Sigma_{ij} \otimes \Sigma_{lk} \otimes \Sigma_{lk} + \Sigma_{ik} \otimes \Sigma_{ik} \otimes \Sigma_{lj} \otimes \Sigma_{lj} \\ & - \Sigma_{ij} \otimes \Sigma_{ik} \otimes \Sigma_{lk} \otimes \Sigma_{lj} - \Sigma_{ik} \otimes \Sigma_{ij} \otimes \Sigma_{lj} \otimes \Sigma_{lk}, \end{aligned} \tag{28}$$

and

$$\Sigma_{ij} = \sigma_y \otimes \sigma_y \otimes e_{ij}. \tag{29}$$

If ρ is defined in $C_{d\times d}$, $\mathcal{A}$ is then defined in $C_{d\times d} \otimes C_{d\times d} \otimes C_{d\times d} \otimes C_{d\times d}$. If the former two subspaces and the latter two ones are regarded as a doubled subspace, respectively. $\mathcal{A}$ can be considered to be defined in $C_{d^2\times d^2} \otimes C_{d^2\times d^2}$. It is easy to find that $\mathcal{A}$ is invariant under the exchange of two doubled subspaces. It is also obvious that $\mathcal{A}$ will be converted to $\mathcal{A}^*$, if the former two subspaces and the latter two ones are exchanged simultaneously. Due to the symmetry, following the analogous procedure to that in Ref. [16], we have the following relations by means of kronecker approximation technique [16,26,27].

$$\mathcal{A} = \sum_{i=1}^{r'} B_i \otimes B_i \tag{30}$$

with B_i defined in $C_{d\times d} \otimes C_{d\times d}$, and

$$B_i = \sum_{j=1}^{r''} (\sigma_i)_j \, (C_i)_j \otimes (C_i)_j^* \tag{31}$$

with $(C_i)_j$ defined in $C_{d\times d}$, and $(\sigma_i)_j$ being the corresponding singular values [28]. Substitute above relations into eq. (25), one can obtain that

$$\tau(\rho) = \inf_U \sum_{i=1}^{N} \left(\sum_{j=1}^{r'} \left(\sum_{m}^{r''} \left| \left(U^T (C_j)_m U \right)_{ii} \right|^2 \right)^2 \right)^{1/4}. \tag{32}$$

Following the procedure of Ref. [16] again, one can also obtain three lower bounds, which have the same form to those in Ref. [16]. Therefore, we do not give these bounds here.

Similarly, one can also find that the numerical realization to calculate the bounds for a mixed state ρ faces the same problem mentioned in Ref. [16], i.e. the lower efficiency of calculation. To avoid the problem, again we employ the method given in Ref. [29] to present an analytic approximation of eq. (32) for weakly mixed states-quasi pure states. In this way, we can conveniently demonstrate the applications of our measure to some quasi pure states.

Analogous to Ref. [29], the tensor $\mathcal{A}$ can be obtained by

$$\begin{aligned}\mathcal{A}^{pm,nq}_{p'm',n'q'} = &\sum_{i\neq l, j\neq k} \sqrt{u_p u_m u_n u_q u_{p'} u_{m'} u_{n'} u_{q'}} \times \left[\left\langle \gamma_p^* \right| \sigma_y \otimes \sigma_y \otimes e_{ij} \left|\gamma_{p'}\right\rangle\right. \\ &\times \left\langle \gamma_m^* \right| \sigma_y \otimes \sigma_y \otimes e_{lk} \left|\gamma_{m'}\right\rangle - \left\langle \gamma_p^* \right| \sigma_y \otimes \sigma_y \otimes e_{ik} \left|\gamma_{p'}\right\rangle \times \left\langle \gamma_m^* \right| \sigma_y \otimes \sigma_y \otimes e_{lj} \left|\gamma_{m'}\right\rangle\Big] \\ &\times \left[\left\langle \gamma_n \right| \sigma_y \otimes \sigma_y \otimes e_{ij} \left|\gamma_{n'}^*\right\rangle \times \left\langle \gamma_q \right| \sigma_y \otimes \sigma_y \otimes e_{lk} \left|\gamma_{q'}^*\right\rangle\right. \\ &\left.- \left\langle \gamma_n \right| \sigma_y \otimes \sigma_y \otimes e_{ik} \left|\gamma_{n'}^*\right\rangle \times \left\langle \gamma_q \right| \sigma_y \otimes \sigma_y \otimes e_{lj} \left|\gamma_{q'}^*\right\rangle\right],\end{aligned} \tag{33}$$

where $|\gamma_\alpha\rangle$ and u_α denote the αth eigenvector and eigenvalue of ρ respectively and all the other quantities are defined similar to those in eq. (9). According to the symmetry of A and the kronecker product approximation technique in above section, A can be formally written as

$$\mathcal{A}^{pm,nq}_{p'm',n'q'} = \sum_\alpha T^\alpha_{pm} \left(T^\alpha_{p'm'}\right)^* T^\alpha_{nq} \left(T^\alpha_{n'q'}\right)^*. \tag{34}$$

The density matrix of quasi pure states has one single eigenvalue μ_1 that is much larger than all the others, which induces a natural order in terms of the small eigenvalues μ_i, $i > 1$. Due to the same reasons to those in Ref. [29], here we consider the second order elements of type $A^{pm,11}_{11,11}$. Therefore, one can have the approximation

$$\mathcal{A}^{pm,nq}_{p'm',n'q'} \simeq \kappa_{pm}\kappa^*_{p'm'}\kappa_{nq}\kappa^*_{n'q'} \ \ with \ \kappa_{pm} = \frac{\mathcal{A}^{pm,11}_{11,11}}{\sqrt[4]{\left(\mathcal{A}^{11,11}_{11,11}\right)^3}}. \tag{35}$$

In this sense, eq. (32) can be simplified significantly:

$$\tau(\rho) \simeq \tau_a(\rho) = \inf_U \sum_i \left|U^T \kappa U\right|_{ii}. \tag{36}$$

$\tau_a(\rho)$ can be given by

$$\tau_a(\rho) = \max\{\lambda_1 - \sum_{i>1} \lambda_i, 0\}, \tag{37}$$

where λ_i is the singular value of κ in decreasing order.

As applications, let us consider two $(2 \times 2 \times 3)$ −dimensional quasi pure states constructed respectively by

$$\rho_1(x) = x\left|GHZ'\right\rangle\left\langle GHZ'\right| + (1-x)\mathbf{1}_{12} \tag{38}$$

and

$$\rho_2(x) = x\left|W'\right\rangle\left\langle W'\right| + (1-x)\mathbf{1}_{12}, \tag{39}$$

where

$$|GHZ'\rangle = \frac{1}{2}(|000\rangle + |101\rangle + |011\rangle + |112\rangle), \tag{40}$$

and

$$|W'\rangle = \frac{1}{\sqrt{3}}\left(|000\rangle + |011\rangle + |112\rangle\right). \tag{41}$$

Note that $|GHZ'\rangle$ and $|W'\rangle$ given in Ref. [11] correspond to GHZ class and W class with high local rank, respectively. The two states can be considered as quasi pure states for $x \geq 0.3$. τ_a for $\rho_1(x)$ and $\rho_2(x)$ are both shown in Fig. 1, where the solid line corresponds to $\tau_a(\rho_1)$ and the dotted line corresponds to $\tau_a(\rho_2)$. Fig. 1 shows the sufficiency to test genuine tripartite entanglement for such quasi pure states. In this sense, the measure presented in the paper can characterize the properties of genuine tripartite entanglement and can serve as an effective indicator of genuine tripartite entanglement.

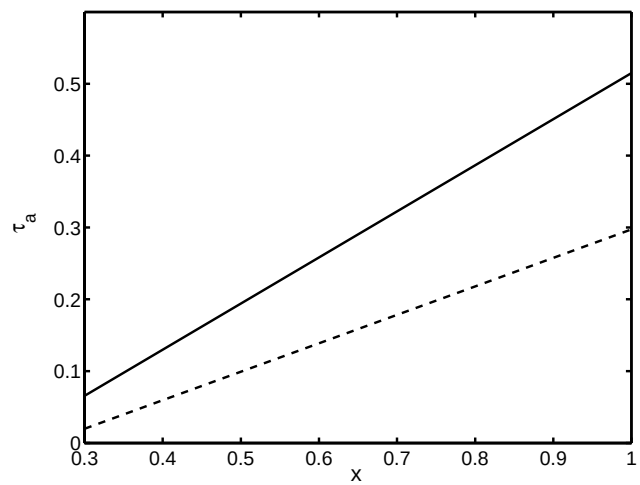

Fig. 1. τ_a (dimensionless) for quasi pure states $\rho_1(x) = x\,|GHZ'\rangle\,\langle GHZ'| + (1-x)\mathbf{1}_{12}$ (solid line) and $\rho_2(x) = x\,|W'\rangle\,\langle W'| + (1-x)\mathbf{1}_{12}$ (dotted line) *vs* x, $x \in [0.3, 1]$.

4 Conclusion and Discussion

In summary, we have introduced an entanglement semi-monotone τ by a new approach to measure the genuine tripartite entanglement existing in a given tripartite $(2 \times 2 \times n)-$dimensional quantum pure states. For $(2 \times 2 \times 2)-$dimensional systems, τ can be reduced to the initial residual entanglement given in Ref. [9], but there exists a neglectable constant difference between them. In particular, it does not vanish for W states with high local rank. In this sense, τ can be considered as a generalization of the initial residual entanglement. What's more, τ can conveniently extended to the case of mixed states by utilizing the kronecker product approximation technique. For the weakly mixed states, i.e. quasi pure states, we have provided an analytic approximation, by which we have investigated the genuine tripartite entanglement of two quasi pure states. Numerical results show that τ obtained from the analytic approximation can serve as an effective indicator of genuine tripartite entanglement

for quasi pure states. Our result can be generalized to $(d \times d \times n)-$dimensional systems, but many details are quite different and the property of entanglement monotone might be lost. We would like to study the generalization in detail elsewhere. What's more, it will be more valuable that τ for pure states can be employed to signal the phase transition of some spin interaction systems by considering tripartite entanglement, which is our forthcoming work.

Acknowledgement

This work was supported by the National Natural Science Foundation of China, under Grant Nos. 10575017 and 60472017.

References

1. M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, 2000).
2. M. Zukowski, A. Zeilinger, M. A. Horne, and A. K. Ekert, Phys. Rev. Lett. **71**, 4287 (1993).
3. C. H. Bennett, et al., Phys. Rev. Lett.**70**,1895 (1993).
4. C. H. Bennett and S. Wiesner, Phys. Rev. Lett. **69**, 2881 (1992).
5. W. K. Wootters, Phys. Rev. Lett. **80**, 2245 (1998).
6. A.Uhlmann, Phys. Rev. A **62**, 032307 (2000).
7. K. Audenaert, F. Verstraete and De Moor, Phys. Rev. A **64**, 052304 (2001).
8. Florian Mintert, Marek Kuś, and Andreas Buchleitner, Phys. Rev. Lett. **92**, 167902 (2004).
9. Valerie Coffman, Joydip Kundu, and William K. Wootters, Phys. Rev. A **61**, 052306 (2000).
10. W. Dür, G. Vidal, and J. I. Cirac, Phys. Rev. A **62**, 062314 (2000).
11. A. Miyake, Phys. Rev. A **67**, 012108 (2003).
12. A. Miyake, F. Verstraete, Phys. Rev. A **69**, 012101 (2004).
13. Andreas Osterloh, Jens Siewert, Phys. Rev. A **72**, 012337 (2005).
14. Alexander Wong and Nelson Christensen, Phys. Rev. A **63**, 044301 (2001).
15. Chang-shui Yu, He-shan Song, Phys. Rev. A **72**, 022333 (2005).
16. Chang-shui Yu, He-shan Song, Phys. Rev. A **73**, 032322 (2006).
17. A. R. R. Carvalho, F. Mintert, A. Buchleitner, Phys. Rev. Lett. **93,** 230501 (2004);
18. Chang-shui Yu, He-shan Song, Phys. Rev. A **73**, 022325 (2006).
19. D. A. Meyer and N. R. Wallach, J. Math. Phys. **43**, 4273 (2002).
20. G. K. Brennen, Quant. Inf. Comp. **3**, 619 (2003).
21. G. Gour, D. A. Meyer, and B. C. Sanders, Phys. Rev. A **72**, 042329 (2005).
22. G. Gour, Phys. Rev. A **72**, 042318 (2005).
23. P. Rungta, V. Bužek, C. M. Caves, et al, Phy. Rev. A **64**, 042315 (2001).
24. W. K. Wootters, Quantum Inf. Comp. **1**, 27 (2001).
25. In fact, the entanglement of W states is a kind of genuine tripartite entanglement [10-12], however, $\tau(W) = 0$ for $(2 \times 2 \times 2)-$dimensional cases. In this sense, the genuine tripartite entanglement mentioned in the paper means those with *nonzero* τ.
26. N. P. Pitsianis, Ph.D. thesis, Cornell University, New York, 1997.
27. C. F. Van Loan and N. P. Pitsianis, in *Linear Algebra for Large Scale and Real Time Applications*, edited by M. S. Moonen and G. H. Golub (Kluwer, Dordrecht, 1993), pp. 293-314.
28. The procedure is the same to that in Ref. [15], hence omitted here. All corresponding parameters are not given explicitly, because they are usually given by numerical processing.
29. Florian Mintert, André R. R. Carvalho, Marek Kuś, and Andreas Buchleitner, Physics Report **415**, 207 (2005).