Corollary 5 *The general solution of $Ly = 0$ is the linear span of the n solutions in Proposition 14.*

We now look at *systems* of equations of the shape

$$Dy_i = \sum_{j=1}^{n} a_{ij} y_j \quad (y_i \text{ in } V). \tag{37}$$

We agree to use the following notation for the derivative of a vector: $D\boldsymbol{y} = (Dy_1, \ldots, Dy_n)$ for $\boldsymbol{y} = (y_1, \ldots, y_n)$, each y_i in V. The system to be solved is

$$D\boldsymbol{y} = A\boldsymbol{y}. \tag{38}$$

If it happens that A is diagonalizable over $\mathbb{C}$,

$$A = PBP^{-1},$$

with P invertible and

$$B = \begin{bmatrix} b_1 & & \\ & \ddots & \\ & & b_n \end{bmatrix},$$

the situation is simple, since

$$D\boldsymbol{y} = PBP^{-1}\boldsymbol{y}, \quad P^{-1}(D\boldsymbol{y}) = B(P^{-1}\boldsymbol{y}).$$

Obviously $P^{-1}(D\boldsymbol{y}) = D(P^{-1}\boldsymbol{y})$. Writing $\boldsymbol{z} = P^{-1}\boldsymbol{y}$,

$$D\boldsymbol{z} = B\boldsymbol{z} \quad , \quad \text{or } Dz_j = b_j z_j \text{ for each } j,$$

which has solution

$$z_j = k_j e^{b_j t}, \quad k_j \text{ in } \mathbb{C}. \tag{39}$$

Now $\boldsymbol{y}$ is given by $\boldsymbol{y} = P\boldsymbol{z}$, $\boldsymbol{z}$ as in (39).

Example 30 Find the general real solution of

$$\begin{aligned} Dy_1 &= 3y_1 - 4y_2 \\ Dy_2 &= y_1 + 3y_2. \end{aligned}$$

Solution. Here $A = \begin{bmatrix} 3 & -4 \\ 1 & 3 \end{bmatrix}$ with characteristic polynomial $(c-3)^2+4$. We proceed to find the *complex* solutions $\boldsymbol{y}$ first.

Eigenspace for $c = 3+2i$. The coefficient matrix for $((3+2i)I - A)\boldsymbol{x} = \mathbf{0}$ is

$$\begin{bmatrix} 2i & 4 \\ -1 & 2i \end{bmatrix} \sim \begin{bmatrix} 1 & -2i \\ 0 & 0 \end{bmatrix} \begin{matrix} \mathrm{I} \times \frac{1}{2i} \\ \mathrm{II} + \mathrm{I}. \end{matrix}$$

A basis of the eigenspace is $(2i, 1)$.

Repeat this calculation with $-i$ in place of i to get a basis $(-2i, 1)$ of the eigenspace for $c = 3-2i$.

Now $A = PBP^{-1}$,

$$P = \begin{bmatrix} 2i & -2i \\ 1 & 1 \end{bmatrix}, \ B = \begin{bmatrix} 3+2i & \\ & 3-2i \end{bmatrix}.$$

The general complex solution is

$$\begin{aligned} \boldsymbol{y} &= \begin{bmatrix} 2i & -2i \\ 1 & 1 \end{bmatrix} \begin{bmatrix} ke^{3+2i)t} \\ \ell e^{(3-2i)t} \end{bmatrix} \quad (k, \ell \text{ in } \mathbb{C}) \\ &= \begin{bmatrix} 2ike^{(3+2i)t} - 2i\ell e^{(3-2i)t} \\ ke^{(3+2i)t} + \ell e^{(3-2i)t} \end{bmatrix}. \end{aligned}$$

To get a real solution we certainly need $y_2 e^{-3t}$ real. That is, we need k_1, k_2, ℓ_1, ℓ_2 to satisfy

$$\mathrm{Im}\{(k_1 + ik_2)(\cos 2t + i\sin 2t) + (\ell_1 + i\ell_2)(\cos 2t - i \sin 2t)\} = 0.$$

Explicitly,

$$(k_1 - \ell_1)\sin 2t + (k_2 + \ell_2)\cos 2t = 0.$$

So $\ell_2 = -k_2, \ell_1 = k_1$, and so $\ell = \bar{k}$. Now, since $z + \bar{z} = 2\mathrm{Re}\, z$,

$$\boldsymbol{y} = 2\mathrm{Re} \begin{bmatrix} 2ike^{(3+2i)t} \\ ke^{(3+2i)t} \end{bmatrix} = \begin{bmatrix} 4e^{3t}(-k_1 \sin 2t - k_2 \cos 2t) \\ 2e^{3t}(k_1 \cos 2t - k_2 \sin 2t) \end{bmatrix}$$

with arbitrary real k_1, k_2.

The method needs modification if A is not diagonalizable over $\mathbb{C}$.

Example 31 Find the general complex solution of

$$\begin{aligned} Dy_1 &= -y_2 \\ Dy_2 &= y_1 - y_2 - y_3 \\ Dy_3 &= y_1 - 2y_3. \end{aligned} \tag{40}$$

Solution. Here $A = \begin{bmatrix} 0 & -1 & 0 \\ 1 & -1 & -1 \\ 1 & 0 & -2 \end{bmatrix}$ has characteristic polynomial

$$\det \begin{bmatrix} c & 1 & 0 \\ -1 & c+1 & 1 \\ -1 & 0 & c+2 \end{bmatrix} = c(c+1)(c+2) + (c+1)$$

(expanding by row 1)

$$= (c+1)(c^2+2c+1) = (c+1)^3.$$

So -1 is the only eigenvalue. The coefficient matrix of $(-I - A)\boldsymbol{x} = \boldsymbol{0}$ is

$$\begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ -1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} \scriptstyle \mathrm{III}-\mathrm{II} \\ \scriptstyle \mathrm{I}\times -1 \\ \scriptstyle \mathrm{II}+\mathrm{I} \end{matrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} \scriptstyle \mathrm{I}-\mathrm{II} \\ \scriptstyle \mathrm{II}\times -1 \end{matrix}.$$

A basis for the eigenspace is $(1, 1, 1)$; A is not diagonalizable. The Jordan form cannot be

$$J_1 = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

since if $AP = PJ_1$ the first and third columns of P are linearly independent eigenvectors of A. So the Jordan form is

$$J_2 = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}.$$

Let $P = [\boldsymbol{v}_1\ \boldsymbol{v}_2\ \boldsymbol{v}_3]$, $AP = PJ_2$. Then $A\boldsymbol{v}_1 = -\boldsymbol{v}_1, A\boldsymbol{v}_2 = \boldsymbol{v}_1 - \boldsymbol{v}_2$, $A\boldsymbol{v}_3 = \boldsymbol{v}_2 - \boldsymbol{v}_3$. Take $\boldsymbol{v}_1 = (1, 1, 1)$. To avoid repetition, we work with the