Exercises for Chapter 5

Reminder: Attempt Exercise a.b after reading Section a.

1.1 Determine the parity of the following permutations.

(i) 1, 2, 4, 3, 6, 5 (ii) 1, 4, 5, 2, 3

(iii) 7, 6, 5, 4, 3, 2, 1 (iv) 1, 3, 5, 7, 2, 4, 6, 8

1.2 Show that any permutation of $1, \ldots, n$ can be obtained, starting with $1, \ldots, n$, by interchanging adjacent pairs of integers at most $\frac{1}{2}\, n(n-1)$ times.

Hint: How many disordered pairs are there in $n, n-1, \ldots, 2, 1$?

1.3 Let A be 2×2. Show that

$$\det(zI - A) = z^2 - (a_{11} + a_{22})z + \det A.$$

1.4 Let A be 2×2 and suppose that $A^2 = 0$. Show that

$$\det(zI - A) = z^2.$$

1.5 Show by an example that it is not true in general that

$$\det(A + B) \le \det A + \det B.$$

2.1 Let B be obtained from the 3×3 matrix A by interchanging row 2 and row 3. By writing out $\det B$ in full, verify that $\det B = -\det A$.

2.2 Evaluate the determinants of the following matrices.

$$\text{(i)} \begin{bmatrix} -2 & 0 & 3 \\ 1 & 0 & -4 \\ 0 & 2 & 6 \end{bmatrix} \qquad \text{(ii)} \begin{bmatrix} -1 & -1 & -2 \\ 3 & 4 & 5 \\ 6 & 8 & 9 \end{bmatrix} \qquad \text{(iii)} \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 1 & 2 \end{bmatrix}.$$

2.3 Given vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_{n-1}$ in $\mathbb{R}^n$, let $T : \mathbb{R}^n \to \mathbb{R}$,

$$T(\boldsymbol{x}) = \det[\boldsymbol{a}_1 \cdots \boldsymbol{a}_{n-1}\ \boldsymbol{x}].$$

(i) Show that T is a linear mapping.

(ii) If $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_{n-1}$ are linearly independent, show that T has rank 1 and write down a basis of Ker T.

2.4 Let A be $n \times n$ of rank r. Show that we can delete $n - r$ rows and $n - r$ columns, leaving a matrix with nonzero determinant. Could we get a similar outcome with fewer deleted rows and columns?

Hint: Delete columns to leave a basis of the column space. What is the dimension of the row space of the remaining $n \times r$ matrix?

3.1 Let A be 3×3. Write out in full the expansion of $\det A$ by row 2. Verify that the expression you obtain is $\det A$.

3.2 Evaluate $\det A$ where

$$A = \begin{bmatrix} 4 & 1 & -2 & -1 \\ -2 & -3 & 0 & -2 \\ 1 & 2 & 0 & 4 \\ 2 & 0 & 1 & 3 \end{bmatrix}$$

(i) by expanding by column 3 and then row 3 of the 3×3 matrices obtained;

(ii) by row reduction.

Which method is quicker?

3.3 Show that the expansion by the first *row* of

$$\det \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ 0 & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

is $a_{11} \det A_{11}$.

4.1 Show that adj $(A^t) = (\text{adj } A)^t$.

4.2 Show that adj $(cA) = c^{n-1}\text{adj } A$.

4.3 Compute adj A if $A = [\boldsymbol{a}_1 \quad k\boldsymbol{a}_1]$ is 2×2, with $\boldsymbol{a}_1 \neq \mathbf{0}$. Verify that in this case
$$\text{nullity (adj } A) + \text{nullity } A = 2.$$

4.4 Let $A = \begin{bmatrix}\boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_1 + \boldsymbol{a}_2\end{bmatrix}$ be 3×3, where $\boldsymbol{a}_1, \boldsymbol{a}_2$ are linearly independent. Compute adj A, and show that
$$\text{nullity (adj } A) + \text{nullity } A = 3.$$

4.5 Let B be an $n \times n$ matrix two of whose rows are zero. Show that adj $B = 0$.

4.6 Apply Cramer's rule to solve the system
$$\begin{aligned} 3x_1 + 2x_2 - x_3 &= 2 \\ x_1 + x_2 \phantom{{}-x_3} &= 5 \\ -x_1 + x_2 - x_3 &= 0. \end{aligned}$$

5.1 Using the formula
$$\det(AB) = \det A \det B,$$
give another proof that a matrix A with determinant 0 is not invertible.

5.2 Show that
$$\det AB = \det A \det B$$
for 2×2 matrices $A = [a_{ij}], B = [b_{ij}]$, by writing out both sides in terms of a_{ij} and b_{ij}.

5.3 Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be linear and let A, B be the matrices of T for the bases $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ and $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_n$ respectively. Show that $\det A = \det B$. The number $\det A$ is said to be the **determinant of** T, written $\det T$.

5.4 Find the area of the parallelogram whose vertices are
$$(0, 0), (1, 6), (2, -5), (3, 1).$$

5.5 Find the area of the parallelogram whose vertices are
$$(1, 1), (3, 4), (4, 8), (6, 11).$$

5.6 Find a formula for the area of the triangle whose vertices are $\mathbf{0}, \boldsymbol{v}_1$ and $\boldsymbol{v}_2$ in $\mathbb{R}^2$.

5.7 Show that the triangle with vertices $(a_1, a_2), (b_1, b_2), (c_1, c_2)$ has area

$$\pm\frac{1}{2}\det\begin{bmatrix} a_1 & a_2 & 1 \\ b_1 & b_2 & 1 \\ c_1 & c_2 & 1 \end{bmatrix}.$$

Hint: Move (a_1, a_2) to the origin by a translation and use Problem 5.6.

5.8 Show that the equation of the line in $\mathbb{R}^2$ through distinct points (a_1, a_2) and (b_1, b_2) can be written

$$\det\begin{bmatrix} a_1 & a_2 & 1 \\ b_1 & b_2 & 1 \\ x_1 & x_2 & 1 \end{bmatrix} = 0.$$

5.9 Let A be a 3×3 invertible matrix whose determinant is an integer and whose column vectors have length $\leq K$. If $A\boldsymbol{x} = \boldsymbol{b}$, show that $|\boldsymbol{x}| \leq \sqrt{3}K^2|\boldsymbol{b}|$.

5.10 Let A be $n \times n$. Show that $\det(A^t A) \geq 0$.

5.11 Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a reflection in the line through $\mathbf{0}$ and $\boldsymbol{v}$. Show that T is linear. Show that $\det T = -1$.

Hint: Choose a basis to make the problem easy.

5.12 Let A and P be invertible matrices. Show that

$$\text{adj }(P^{-1}AP) = P^{-1}(\text{adj } A)P.$$

Is this formula correct if P is invertible and A is not?

5.13 Show that

$$\det\begin{bmatrix} 1 & x & x^3 \\ 1 & y & y^3 \\ 1 & z & z^3 \end{bmatrix} = (y-x)(z-x)(z-y)(x+y+z).$$

Hint: Use two row operations to simplify, and extract factors from rows where possible.

5.14 Produce a factorization similar to Exercise 5.13 for

$$\det \begin{bmatrix} 1 & x^2 & x^3 \\ 1 & y^2 & y^3 \\ 1 & z^2 & z^3 \end{bmatrix}.$$

5.15 Use the Vandermonde determinant to show that, given distinct real $x_1, \ldots, x_n$ and any numbers $c_1, \ldots, c_n$, there is a polynomial $P(x) = a_0x^{n-1}+a_1x^{n-2}+\cdots+a_{n-2}x+a_{n-1}$ such that $P(x_1) = c_1, \ldots, P(x_n) = c_n$.