Proof. Both $(AB)^t$ and B^tA^t are $p \times m$ matrices. The i, j entry of AB is

$$\sum_{k=1}^{n} a_{ik}b_{kj}.$$

Thus the i, j entry of $(AB)^t$ is $\sum_{k=1}^{n} a_{jk}b_{ki}$. This is the inner product of the i-th row $(b_{1i}, \dots, b_{ni})$ of B^t with the j-th column $(a_{j1}, \dots, a_{jn})$ of A^t, which is the i, j entry of B^tA^t.

Proposition 14 *Let A be an invertible matrix. Then A^t is invertible and*

$$(A^t)^{-1} = (A^{-1})^t.$$

Proof. By Proposition 13,

$$(A^{-1})^tA^t = (AA^{-1})^t = I^t = I,$$

and similarly,

$$A^t(A^{-1})^t = (A^{-1}A)^t = I.$$

This proves that $(A^{-1})^t$ is the inverse of A^t.

6 Change of basis

Let $\boldsymbol{v}_1, \dots, \boldsymbol{v}_n$ and $\boldsymbol{w}_1, \dots, \boldsymbol{w}_n$ be two bases of $\mathbb{R}^n$. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be linear. If $T\left(\sum_{j=1}^{n} y_j\boldsymbol{v}_j\right)$ has coordinate vector $A\boldsymbol{y}$ for the basis $\boldsymbol{v}_1, \dots, \boldsymbol{v}_n$, we say that **the matrix of** T is A **for the basis** $\boldsymbol{v}_1, \dots, \boldsymbol{v}_n$. 'The matrix of T is A' now becomes a shorthand for 'The matrix of T is A for the standard basis.'

Proposition 15 (Change of basis). *Let T have matrix A for the basis $\boldsymbol{v}_1, \dots, \boldsymbol{v}_n$. Suppose*

$$\boldsymbol{y} = P\boldsymbol{z}, \tag{17}$$

where $\boldsymbol{y}$ and $\boldsymbol{z}$ are coordinate vectors for $\boldsymbol{v}_1, \dots, \boldsymbol{v}_n$ and $\boldsymbol{w}_1, \dots, \boldsymbol{w}_n$ respectively. Then T has matrix $P^{-1}AP$ for the basis $\boldsymbol{w}_1, \dots, \boldsymbol{w}_n$. In particular $P = [\boldsymbol{w}_1 \cdots \boldsymbol{w}_n]$ if $\boldsymbol{v}_1, \dots, \boldsymbol{v}_n$ is the standard basis.

Proof. The change of coordinates can also be written

$$\boldsymbol{z} = P^{-1}\boldsymbol{y}$$

(multiply both sides of (17) by P^{-1}). The coordinate vector of $T\Big(\sum_{i=1}^{n} z_i\boldsymbol{w}_i\Big)$ is $A(P\boldsymbol{z}) = AP\boldsymbol{z}$ for $\boldsymbol{v}_1, \dots, \boldsymbol{v}_n$. Hence the coordinate vector of $T\Big(\sum_{i=1}^{n} z_i\boldsymbol{w}_i\Big)$ is $P^{-1}(AP\boldsymbol{z})$ for $\boldsymbol{w}_1, \dots, \boldsymbol{w}_n$. Since

$$P^{-1}(AP\boldsymbol{z}) = (P^{-1}AP)\boldsymbol{z},$$

this proves the proposition. (The final sentence is simply a reminder of a fact from Proposition 13 of Chapter 3.)

In Chapter 6 we shall see that the matrix of T for a suitably chosen basis can have a much simpler appearance than the matrix of T for the standard basis.

Example 19 Find the matrix of the mapping

$$T(\boldsymbol{x}) = (2x_1 - x_2, x_1 - 4x_2)$$

for the basis $\boldsymbol{w}_1 = (1,2), \boldsymbol{w}_2 = (2,5)$.

Solution. T has matrix $A = \begin{bmatrix} 2 & -1 \\ 1 & -4 \end{bmatrix}$ for the basis $\boldsymbol{e}_1, \boldsymbol{e}_2$. The matrix B of T for $\boldsymbol{w}_1, \boldsymbol{w}_2$ is $P^{-1}AP$ where $P = [\boldsymbol{w}_1\ \boldsymbol{w}_2]$. So

$$\begin{aligned} B &= \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}^{-1} \begin{bmatrix} 2 & -1 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 8 & 3 \\ -3 & -2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 14 & 31 \\ -7 & -16 \end{bmatrix}. \end{aligned}$$

Thus

$$T(z_1\boldsymbol{w}_1 + z_2\boldsymbol{w}_2) = (14z_1 + 31z_2)\boldsymbol{w}_1 + (-7z_1 - 16z_2)\boldsymbol{w}_2$$

(check this for $\boldsymbol{z} = (1,0), (0,1)$!).