3 Subspaces of $\mathbb{R}^n$

Definition 3 Let V be a set of vectors in $\mathbb{R}^n$, *not* the empty set. V is a **subspace** of $\mathbb{R}^n$ if

(7) $$a\boldsymbol{u} + b\boldsymbol{v} \text{ is in } V \text{ for all } \boldsymbol{u} \text{ and } \boldsymbol{v} \text{ in } V \text{ and real } a, b.$$

We call the property (7) the **closure property** .

The set consisting of $\mathbf{0}$ alone is a subspace. We call it the **zero subspace**. We write $\mathbf{0}$ for this subspace. This is not good 'grammar' since $\mathbf{0}$ is a vector rather than a set of vectors, but it is convenient.

A subspace in general turns out to be the linear span of some set.

Proposition 3 *Let V be a set of vectors in $\mathbb{R}^n$. The following two assertions are equivalent.*

(i) V is a subspace of $\mathbb{R}^n$.

(ii) $V = Span\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_k\}$ for some $\boldsymbol{v}_1, \dots, \boldsymbol{v}_k$ in $\mathbb{R}^n$.

Proof. Suppose that (i) holds. If V is the zero subspace, then $V = \text{Span}\{\mathbf{0}\}$ and we get (ii). If V is not the zero subspace, pick $\boldsymbol{v}_1 \neq \mathbf{0}$ in V. Of course $\text{Span}\{\boldsymbol{v}_1\}$ is a subset of V because of (7) (with $\boldsymbol{v}_1 = \boldsymbol{u}, \boldsymbol{v}_2 = \mathbf{0}$). If there are vectors in V, not in $\text{Span}\{\boldsymbol{v}_1\}$, pick $\boldsymbol{v}_2$ in V, $\boldsymbol{v}_2$ not in $\text{Span}\{\boldsymbol{v}_1\}$.

We continue this game in the following way. Once we have got k vectors $\boldsymbol{v}_1, \dots, \boldsymbol{v}_k$ in V, pick $\boldsymbol{v}_{k+1}$ in V but not in $\text{Span}\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_k\}$ *if possible.* If this is not possible, stop the game. The game must stop after at most n steps, because Proposition 1 shows that $\boldsymbol{v}_1, \dots, \boldsymbol{v}_k$ are independent, which is impossible with $k = n+1$. When we stop with $\boldsymbol{v}_1, \dots, \boldsymbol{v}_k$, then $\text{Span}\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_k\}$ is a subset of V because of (7). $\text{Span}\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_k\}$ is, in fact, equal to V, otherwise the game would continue. Thus (ii) holds.

Now suppose that (ii) holds. Take any $\boldsymbol{u}$ and $\boldsymbol{v}$ in V and real a, b. Then

$$\boldsymbol{u} = a_1\boldsymbol{v}_1 + \dots + a_k\boldsymbol{v}_k, \boldsymbol{v} = b_1\boldsymbol{v}_1 + \dots + b_k\boldsymbol{v}_k$$

for some real a_i and b_i;

$$a\boldsymbol{u} + b\boldsymbol{v} = (aa_1 + bb_1)\boldsymbol{v}_1 + \dots + (aa_k + bb_k)\boldsymbol{v}_k.$$

So $a\boldsymbol{u} + b\boldsymbol{v}$ is in V, and (i) holds.

For the subspace in the following example, the closure property is easier to perceive than the property (ii) of Proposition 3.

Example 11 The solution set U of a homogeneous linear system with $m \times n$ coefficient matrix A,

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= 0 \\ &\cdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &= 0 \end{aligned}$$

is a subspace of $\mathbb{R}^n$. We say that U is the **null space** of A, and write,

$$U = \text{Nul } A.$$

To see that U is a subspace, take $\boldsymbol{x}$ and $\boldsymbol{y}$ in U. Then $\boldsymbol{a}_i \cdot \boldsymbol{x} = 0 = \boldsymbol{a}_i \cdot \boldsymbol{y}$, where $\boldsymbol{a}_i = (a_{i1}, \dots, a_{in})$. Consequently $\boldsymbol{a}_i \cdot (a\boldsymbol{x} + b\boldsymbol{y}) = 0$. Here i is any of $1, \dots, m$. This shows that $a\boldsymbol{x} + b\boldsymbol{y}$ is in U, and U is a subspace.

We do, in fact, know how to write down $\boldsymbol{v}_1, \dots, \boldsymbol{v}_k$ in U which satisfy $U = \text{Span}\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_k\}$. In Chapter 2 we gave the general solution of the system in the form

$$x_{m(1)}\boldsymbol{v}_1 + \cdots + x_{m(k)}\boldsymbol{v}_k$$

where $x_{m(1)}, \dots, x_{m(k)}$ are free variables. This is equivalent to

$$U = \text{Span}\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_k\}.$$

For example, just after Proposition 4 we noted that with A as in Example 10, the general solution of $A\boldsymbol{x} = \boldsymbol{0}$ is

$$x_3\boldsymbol{v}_1 + x_4\boldsymbol{v}_2, \boldsymbol{v}_1 = (-5, 1, 1, 0, 0), \boldsymbol{v}_2 = (2, 3, 0, 1, 0).$$

Here the null space is $U = \text{Span}\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$.

The only subspaces of $\mathbb{R}^3$ are the zero subspace, lines through $\boldsymbol{0}$, planes through $\boldsymbol{0}$, and $\mathbb{R}^3$ itself. The last three alternatives correspond to the game in Proposition 3 stopping when $k = 1, 2, 3$ respectively.

The game we used in the proof of Proposition 3 generates a set $\boldsymbol{v}_1, \dots, \boldsymbol{v}_k$ that is a **basis**, as defined below.

Definition 4 Let V be a subspace of $\mathbb{R}^n$, not the zero subspace. A **basis** of V is a linearly independent set $\boldsymbol{v}_1, \dots, \boldsymbol{v}_k$ such that

$$V = \text{Span}\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_k\}.$$

For any nonzero subspace V of $\mathbb{R}^n$, the set $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k$ obtained in the proof of Proposition 3 is linearly independent. This follows from Proposition 1, since $\boldsymbol{v}_j$ is not a combination of $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{j-1}$. Accordingly this set $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k$ is a basis of V.

Of course, there are infinitely many choices of basis. For example, if V is a plane in $\mathbb{R}^3$, the last paragraph assures us that *any* nonproportional pair of vectors $\boldsymbol{v}_1, \boldsymbol{v}_2$ in V could be chosen as a basis of V.

A crucial observation is that any two bases of V have the same number of vectors. ('Bases,' pronounced bayseas, is the plural of basis.)

Proposition 4 *Let V be a subspace of $\mathbb{R}^n$. Any two bases of V have the same number of vectors.*

Proof. Let $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_k$ and $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_j$ be two bases of V. We have j linearly independent elements $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_j$ in $V = \mathrm{Span}\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_k\}$. Proposition 2 assures us that $j \leq k$. Reversing roles, we get $k \leq j$. So $k = j$.

Proposition 4 enables us to define dimension.

Definition 5 The **dimension** of a subspace V of $\mathbb{R}^n$ is the number of vectors in any basis of V. We write $\dim V$ for the dimension of V.

Naturally $\dim V \leq n$, since $\mathbb{R}^n$ cannot contain $n+1$ linearly independent vectors.

For completeness, the zero subspace is assigned dimension 0. The zero subspace does not have a basis.

We can now deduce that the dimension of $\mathbb{R}^n$ is n. For the set $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$ in Example 4 is independent. The vector equation $x_1\boldsymbol{e}_1 + \cdots + x_n\boldsymbol{e}_n = \mathbf{0}$ reads $x_1 = 0, x_2 = 0, \ldots, x_n = 0$. So $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$ is one basis of $\mathbb{R}^n$; we call it the **standard basis** . Now we know, of course, that

$$\dim \mathbb{R}^n = n.$$

The dimension of a plane V through $\mathbf{0}$ in $\mathbb{R}^3$ is 2; we noted a few paragraphs ago that we can find (many) bases of V with 2 elements. The dimension of a line through $\mathbf{0}$ in $\mathbb{R}^n$ is 1.

It seems reasonable that the dimension of a hyperplane W in $\mathbb{R}^n$, with equation

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0, \tag{8}$$

should be $n-1$. The easiest demonstration is to provide a basis. Suppose, for instance, that $a_1 \neq 0$. The general solution of the linear system (8) is

$$x_1 = \left(-\frac{a_2}{a_1}\right) x_2 + \cdots + \left(-\frac{a_n}{a_1}\right) x_n$$

or

$$\text{(9)} \qquad \boldsymbol{x} = x_2 \begin{bmatrix} -a_2/a_1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \cdots + x_n \begin{bmatrix} -a_n/a_1 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} = x_2\boldsymbol{v}_2 + \cdots + x_n\boldsymbol{v}_n,$$

say. Here $x_2, \ldots, x_n$ are free. The hyperplane W is Span$\{\boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\}$. Now $\boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ are independent. To see this, set $x_2\boldsymbol{v}_2 + \cdots + x_n\boldsymbol{v}_n = \mathbf{0}$ in (9). We get a linear system which includes the equations $x_2 = 0, x_3 = 0, \ldots, x_n = 0$. Therefore $\boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ is a basis of W, and

$$\dim W = n - 1.$$

This suggests a general result on the dimension of the space Nul A. The general solution to

$$A\boldsymbol{x} = \mathbf{0}$$

is

$$\text{(10)} \qquad \boldsymbol{v} = x_{m(1)}\boldsymbol{v}_1 + \cdots + x_{m(k)}\boldsymbol{v}_k \qquad (x_{m(1)}, \ldots, x_{m(k)} \text{ free})$$

as we recalled in Example 5. Now k of the coordinates of $\boldsymbol{v}$ are actually $x_{m(1)}, \ldots, x_{m(k)}$. For instance, let $n = 5$, and suppose that the general solution is

$$\boldsymbol{v} = \begin{bmatrix} -2x_2 - 3x_3 - x_5 \\ x_2 \\ x_3 \\ 4x_2 + 6x_3 + 2x_5 \\ x_5 \end{bmatrix} = x_2\boldsymbol{v}_1 + x_3\boldsymbol{v}_2 + x_5\boldsymbol{v}_3$$

(x_2, x_3, x_5 free). Here $\boldsymbol{v}_1 = (-2, 1, 0, 4, 0)$, $\boldsymbol{v}_2 = (-3, 0, 1, 6, 0)$, $\boldsymbol{v}_3 = (-1, 0, 0, 2, 1)$. Three coordinates of $\boldsymbol{v}$ are x_2, x_3, x_5. If $x_2\boldsymbol{v}_1 + x_3\boldsymbol{v}_2 + x_5\boldsymbol{v}_3$ is $\mathbf{0}$, then x_2, x_3, x_5 are 0. In the general case, if we set $\boldsymbol{v}$ equal to $\mathbf{0}$ in (10), then $x_{m(1)}, \ldots, x_{m(k)}$ are 0. It follows that $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k$ not only have span equal to Nul A, but are linearly independent. Since $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k$ is a basis of Nul A, we have established:

Proposition 5 dim(*Nul* A) *is the number of non-pivot columns of* A.

Here we use the simple fact that there is a free variable for each non-pivot column.

The **nullity** of A, or nullity A, is another expression used for dim(Nul A).

Example 12 Find a basis of Nul A, and determine the nullity of A, where

$$A = \begin{bmatrix} 1 & 0 & 1 & 2 & 0 & 0 \\ -4 & 1 & 1 & -11 & 0 & -1 \\ -7 & 0 & -7 & -14 & 1 & -2 \\ 6 & 0 & 6 & 12 & 0 & 1 \end{bmatrix}.$$

Solution. You will find that the reduced echelon form of A is

$$B = \begin{bmatrix} 1 & 0 & 1 & 2 & 0 & 0 \\ 0 & 1 & 5 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The general solution of $A\boldsymbol{x} = \mathbf{0}$, in which x_3 and x_4 are free variables, is

$$\begin{bmatrix} -x_3 - 2x_4 \\ -5x_3 + 3x_4 \\ x_3 \\ x_4 \\ 0 \\ 0 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ -5 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 3 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = x_3\boldsymbol{v}_1 + x_4\boldsymbol{v}_2, \text{ say.}$$

Now $\boldsymbol{v}_1, \boldsymbol{v}_2$ is a basis of Nul A. The linear independence of $\boldsymbol{v}_1, \boldsymbol{v}_2$ is obvious. By definition, the nullity of A is 2.

4 General propositions about bases

The following propositions are very general and useful.

Proposition 6 *Let* $\boldsymbol{v}_1, \dots, \boldsymbol{v}_j$ *be linearly independent vectors in a subspace* V *of* $\mathbb{R}^n$. *Suppose* $\dim V = r$. *We can find* $r - j$ *vectors* $\boldsymbol{v}_{j+1}, \dots \boldsymbol{v}_r$ *to obtain a basis* $\boldsymbol{v}_1, \dots, \boldsymbol{v}_j, \boldsymbol{v}_{j+1}, \dots, \boldsymbol{v}_r$ *of* V.