There is a simple relationship between the solution of a given inhomogeneous system

$$A\boldsymbol{x} = \boldsymbol{b}$$

and the homogeneous system with the same coefficient matrix,

$$A\boldsymbol{x} = \mathbf{0}.$$

We can illustrate this by looking back at Example 10. The general solution of $A\boldsymbol{x} = \boldsymbol{b}$ is

$$(3, 4, 0, 0, 8) + x_3(-5, 1, 1, 0, 0) + x_4(2, 3, 0, 1, 0) \quad (x_3, x_4 \text{ free}).$$

The general solution of $A\boldsymbol{x} = \mathbf{0}$ is

$$x_3(-5, 1, 1, 0, 0) + x_4(2, 3, 0, 1, 0).$$

In other words, for the general solution of $A\boldsymbol{x} = \boldsymbol{b}$ we take the general solution of $A\boldsymbol{x} = \mathbf{0}$ and add a particular solution of $A\boldsymbol{x} = \boldsymbol{b}$.

Proposition 5 *Let $A\boldsymbol{x} = \boldsymbol{b}$ be a linear system, which has a solution $\boldsymbol{x} = \boldsymbol{d}$. The general solution is $\boldsymbol{d} + \boldsymbol{y}$, where $\boldsymbol{y}$ is the general solution of the system*

$$A\boldsymbol{x} = \mathbf{0}.$$

Proof. Since $A\boldsymbol{d} = \boldsymbol{b}$ and

$$A(\boldsymbol{y} + \boldsymbol{d}) = A\boldsymbol{y} + A\boldsymbol{d} = A\boldsymbol{y} + \boldsymbol{b},$$

the equation $A\boldsymbol{y} = \mathbf{0}$ is equivalent to the equation $A(\boldsymbol{y} + \boldsymbol{d}) = \boldsymbol{b}$. This is just a rephrasing of the result that we are to prove.

6 Electric circuits

Figure 1 provides information about an electric circuit. Voltage sources are shown like this: $|\,|$. The short line indicates the negative terminal. A voltage source drives electric current away from the positive terminal (the long line). The **resistance** of a piece of a circuit is shown next to a symbol . The notation $h\Omega$ indicates a resistance of h ohms, which means that a voltage source of k volts would send k/h amperes (amps) of current through that piece of circuit considered by itself. The I_j are the currents in the loop inside which they are written. A positive I_j indicates current flowing anticlockwise; a negative I_j thus means a clockwise current. Some loops share a section of circuit, and we interpret this as follows in Figure 1, for example. The current $I_1 - I_2$ flows from left to right in the section common to I_1 and I_2. The current $I_4 - I_3$ flows downwards in the section common to I_3 and I_4. It was determined by Kirchoff that in a given loop, the **algebraic sum of 'resistance × current' products equals the algebraic sum of voltage sources**. The term 'algebraic sum' means that we use the sign convention above for currents; similarly, we consider voltage sources that drive current anticlockwise as positive.

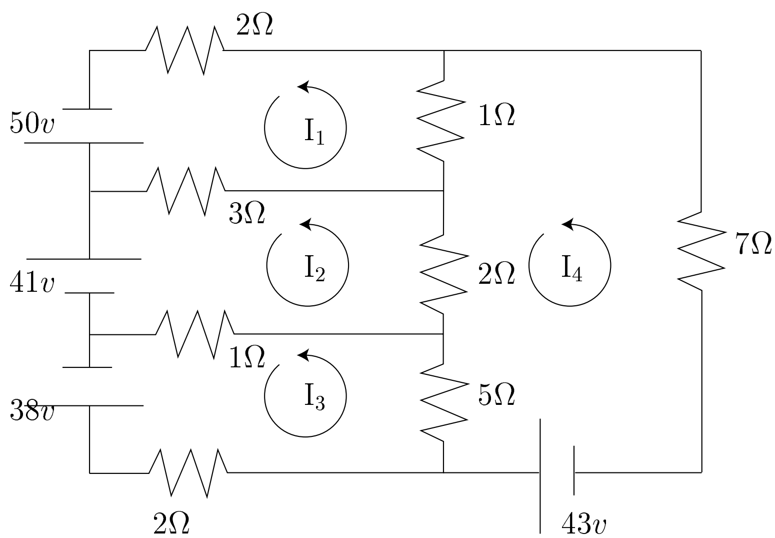

Figure 1. The circuit for Example 12.

Example 12 Consider the circuit in Figure 1. With a little thought we reach the following linear system for $\boldsymbol{I} = (I_1, I_2, I_3, I_4)$.

$$\begin{aligned} 6I_1 - 3I_2 \qquad\quad - \quad I_4 &= 50 \\ -3I_1 + 6I_2 - \ I_3 - \ 2I_4 &= -41 \\ -I_2 + 8I_3 - \ 5I_4 &= 38 \\ -I_1 - 2I_2 - 5I_3 + 15I_4 &= -43. \end{aligned}$$

The augmented matrix is

$$\begin{bmatrix} 6 & -3 & 0 & -1 & 50 \\ -3 & 6 & -1 & -2 & -41 \\ 0 & -1 & 8 & -5 & 38 \\ -1 & -2 & -5 & 15 & -43 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 5 & -15 & 43 \\ 0 & 12 & 14 & -47 & 88 \\ 0 & 1 & -8 & 5 & -38 \\ 0 & -15 & -30 & 89 & -208 \end{bmatrix} \begin{array}{l} \mathrm{I} \leftrightarrow \mathrm{IV} \\ \mathrm{I} \times -1 \\ \mathrm{II} + 3\mathrm{I} \\ \mathrm{IV} - 6\mathrm{I} \\ \mathrm{III} \times -1 \end{array}$$

$$\sim \begin{bmatrix} 1 & 2 & 5 & -15 & 43 \\ 0 & 1 & -8 & 5 & -38 \\ 0 & 0 & 110 & -107 & 544 \\ 0 & 0 & -150 & 164 & -778 \end{bmatrix} \begin{array}{l} \mathrm{II} \leftrightarrow \mathrm{III} \\ \mathrm{III} - 12\mathrm{II} \\ \mathrm{IV} + 15\mathrm{II} \end{array} \sim \begin{bmatrix} 1 & 2 & 5 & -15 & 43 \\ 0 & 1 & -8 & 5 & -38 \\ 0 & 0 & 110 & -107 & 544 \\ 0 & 0 & 0 & 1 & -2 \end{bmatrix} \begin{array}{l} \mathrm{IV} + \frac{15}{11}\,\mathrm{III} \\ \mathrm{IV} \times \frac{11}{199} \end{array}$$

$$\sim \begin{bmatrix} 1 & 2 & 5 & 0 & 13 \\ 0 & 1 & -8 & 0 & -28 \\ 0 & 0 & 110 & 0 & 330 \\ 0 & 0 & 0 & 1 & -2 \end{bmatrix} \begin{array}{l} \mathrm{I} + 15\mathrm{IV} \\ \mathrm{II} - 5\mathrm{IV} \\ \mathrm{III} + 107\mathrm{IV} \end{array} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 6 \\ 0 & 1 & 0 & 0 & -4 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & -2 \end{bmatrix} \begin{array}{l} \mathrm{III} \times \frac{1}{110} \\ \mathrm{II} + 8\mathrm{III} \\ \mathrm{I} - 5\mathrm{III} \\ \mathrm{I} - 2\mathrm{II} \end{array}$$

so that $\boldsymbol{I} = (6, -4, 3, -2)$. Our physical intuition certainly leads us to expect a unique solution, and this has been confirmed.

Example 13 The augmented matrix associated with Figure 2 is

$$\begin{bmatrix} 9 & -4 & -2 & 70 \\ -4 & 5 & -1 & -26 \\ -2 & -1 & 10 & -27 \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{1}{2} & -5 & \frac{27}{2} \\ 0 & 7 & -21 & 28 \\ 0 & -\frac{17}{2} & 43 & -\frac{103}{2} \end{bmatrix} \begin{array}{l} \mathrm{I} \leftrightarrow \mathrm{III} \\ \mathrm{I} \times -\frac{1}{2} \\ \mathrm{II} + 4\mathrm{I} \\ \mathrm{III} - 9\mathrm{I} \end{array}$$

$$\sim \begin{bmatrix} 1 & \frac{1}{2} & -5 & \frac{27}{2} \\ 0 & 1 & -3 & 4 \\ 0 & 0 & \frac{35}{2} & -\frac{35}{2} \end{bmatrix} \begin{array}{l} \mathrm{II} \times \frac{1}{7} \\ \mathrm{III} + \frac{17}{2}\,\mathrm{II} \end{array} \sim \begin{bmatrix} 1 & 0 & 0 & 8 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{array}{l} \mathrm{III} \times \frac{2}{35} \\ \mathrm{II} + 3\mathrm{III} \\ \mathrm{I} + 5\mathrm{III} \\ \mathrm{I} - \frac{1}{2}\,\mathrm{II} \end{array}$$

so that $\boldsymbol{I} = (8, 1, -1)$.

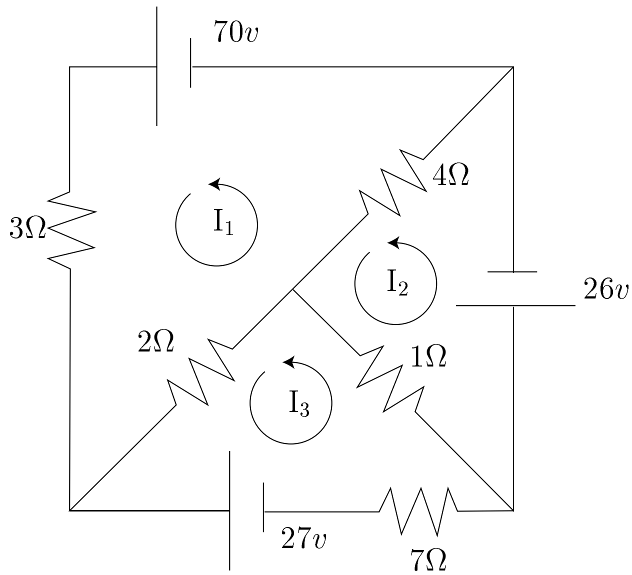

Figure 2. The circuit for Example 13.