or about 6.1 kilometers. The distance is easily seen to be initially decreasing, so that the minimum distance occurs for a positive value of \( t \).

3 Three dimensional Euclidean space \( \mathbb{R}^3 \)

Consider a third directed segment \( e_3 \) of length 1 along with the pair \( e_1, e_2 \) discussed in Section 1, which is orthogonal to both \( e_1, e_2 \). We can now label any point in space as \( (x_1, x_2, x_3) \); this is the point reached by starting at the origin, moving distance \( x_1 \) (with appropriate interpretation according to sign) in direction \( e_1 \), then \( x_2 \) in direction \( e_2 \) and then \( x_3 \) in direction \( e_3 \). See Figure 16.

![Figure 16. Points of three-dimensional space as ordered triples.](image)

Think of \( e_1, e_2 \) as pointing east and north on the (flat) earth and \( e_3 \) as pointing skywards. Now identify points in space with triples of real numbers to get \( \mathbb{R}^3 \). Formally, \( \mathbb{R}^3 \) is the set of ordered triples \( \mathbf{x} = (x_1, x_2, x_3) \) with each \( x_i \) real. Again, we do not distinguish points from directed segments from \( \mathbf{0} \) to \( \mathbf{x} \), so \( \mathbb{R}^3 \) consists of vectors; and again, we may not always distinguish \( \mathbf{x} \) from the translate whose initial point is \( \mathbf{a} = (a_1, a_2, a_3) \) and terminal point \( (x_1 + a_1, x_2 + a_2, x_3 + a_3) \) (Figure 17). A double use of Pythagoras’s theorem, for the triangles with vertices \( \mathbf{0}, (0, x_2, 0), (x_1, x_2, 0) \), and vertices \( \mathbf{0}, (x_1, x_2, 0), (x_1, x_2, x_3) \) leads to the formula for the length \( |\mathbf{x}| \) of vector \( \mathbf{x} \) (Figure 18):

\[
|\mathbf{x}| = (x_1^2 + x_2^2 + x_3^2)^{1/2}.
\]

For, in the second triangle, \( |\mathbf{x}|^2 = ((x_1^2 + x_2^2)^{1/2})^2 + x_3^2 = x_1^2 + x_2^2 + x_3^2 \).
Definition 5 The sum of \( a = (a_1, a_2, a_3) \) and \( b = (b_1, b_2, b_3) \) is
\[
a + b = (a_1 + b_1, a_2 + b_2, a_3 + b_3).
\]

Geometrically, \( a + b \) is at the terminal point of the vector \( b \) if \( b \) is placed with initial point at \( a \). Alternatively, the description of \( a + b \), as the fourth vertex of the parallelogram with vertices 0, \( a, b \), that we used in \( \mathbb{R}^2 \), is valid. This is often referred to as the parallelogram law of addition.
Subtraction of \( z \) from \( y \) is defined by

\[
(y - z) + z = y.
\]

Thus

\[
y - z = (y_1 - z_1, y_2 - z_2, y_3 - z_3).
\]

It is clear that the distance from \( z \) to \( y \) is \( |y - z| \).

**Definition 6** The **scalar product** \( cx \), where \( c \) is real and \( x \) is in \( \mathbb{R}^3 \), is

\[
cx = (cx_1, cx_2, cx_3).
\]

We can, as before, describe \( cx \) as the vector pointing in the same direction as \( x \), but with length \( c \) times that of \( x \), if \( c > 0 \); or the opposite direction, with length \(-c\) times that of \( x \), if \( c < 0 \). This depends on the formula

\[
|cx| = |c| |x|
\]

which we can get in a very similar way to the proof in \( \mathbb{R}^2 \):

\[
|cx| = ((cx_1)^2 + (cx_2)^2 + (cx_3)^2)^{1/2} = |c|(|x_1|^2 + |x_2|^2 + |x_3|^2)^{1/2} = |c| |x|.
\]

You can probably guess the formula that we use to define inner product.

It is

**Definition 7** The **inner product** of \( u \) and \( v \) is

\[
u \cdot v = u_1v_1 + u_2v_2 + u_3v_3.
\]

This is consistent with Definition 4 if we think of \( \mathbb{R}^2 \) as being the same as the set of all \((x_1, x_2, 0)\) (or, the set of all \(x_1e_1 + x_2e_2\)) where \( x_1 \) and \( x_2 \) are real.

Note that the equation (4) from \( \mathbb{R}^2 \),

\[
|u| = (u \cdot u)^{1/2},
\]

holds good in \( \mathbb{R}^3 \).

The angle \( \alpha \) between nonzero vectors \( u, v \) can be defined by drawing a plane that contains \( \mathbf{0}, u, v \) and measuring \( \alpha \) (between 0 and \( \pi \)) in that plane.
Proposition 5  We have

\[ u \cdot v = |u| \, |v| \cos a. \]

Proof.  We may repeat the proof of Proposition 3 verbatim.

‘Verbatim’ means ‘word for word.’  When you want to use for yourself a
proof along these lines, bear in mind that students regularly arrive at wrong
conclusions this way.  If you try it with Proposition 4, you arrive at the false
conclusion that

\[ u \cdot x = u \cdot a \]

is the equation of a line in \( \mathbb{R}^3 \) (it is a plane; see below!).  Thus you have to
be very careful that arguments still work in a new context.

As in \( \mathbb{R}^2 \), for nonzero vectors \( u \) and \( v \) the relation

\[ (9) \quad u \cdot v = 0 \]

is equivalent to \( u, v \) being perpendicular.  We say \( u \) and \( v \) are orthogonal
if (9) holds.

Example 8  The angle between the vectors \((1, 7, b)\) and \((-2, 2, 1)\) is \( a = \cos^{-1}(1/3) \).  Find \( b \).

Solution.  We know that

\[
(1, 7, b) \cdot (-2, 2, 1) = |(1, 7, b)| \, |(-2, 2, 1)| \cos a \\
= |(1, 7, b)| \, |(-2, 2, 1)| \frac{1}{3}.
\]

Thus

\[-2 + 14 + b = (1^2 + 7^2 + b^2)^{1/2} \frac{1}{3} (2^2 + 2^2 + 1)^{1/2} 1/3; \]

\[
b + 12 = (b^2 + 50)^{1/2}.
\]

If we square both sides and cancel \( b^2 \) we find that

\[ 24b + 144 = 50; \ b = -47/12. \]