or about 6.1 kilometers. The distance is easily seen to be initially decreasing, so that the minimum distance occurs for a positive value of t.

3 Three dimensional Euclidean space $\mathbb{R}^3$

Consider a third directed segment $\boldsymbol{e}_3$ of length 1 along with the pair $\boldsymbol{e}_1, \boldsymbol{e}_2$ discussed in Section 1, which is orthogonal to both $\boldsymbol{e}_1, \boldsymbol{e}_2$. We can now label any point in space as (x_1, x_2, x_3): this is the point reached by starting at the origin, moving distance x_1 (with appropriate interpretation according to sign) in direction $\boldsymbol{e}_1$, then x_2 in direction $\boldsymbol{e}_2$ and then x_3 in direction $\boldsymbol{e}_3$. See Figure 16.

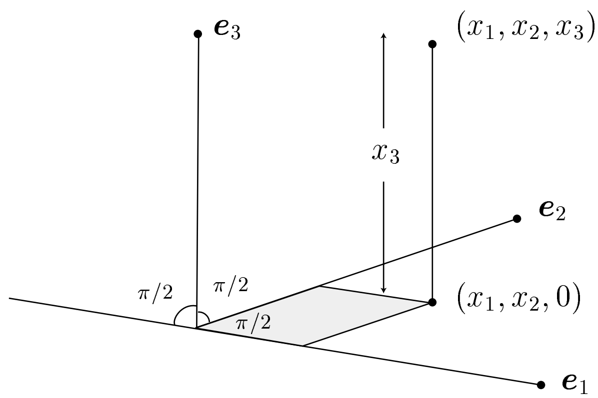

Figure 16. Points of three-dimensional space as ordered triples.

Think of $\boldsymbol{e}_1, \boldsymbol{e}_2$ as pointing east and north on the (flat) earth and $\boldsymbol{e}_3$ as pointing skywards. Now identify points in space with triples of real numbers to get $\mathbb{R}^3$. Formally, $\mathbb{R}^3$ is the set of ordered triples $\boldsymbol{x} = (x_1, x_2, x_3)$ with each x_i real. Again, we do not distinguish points from directed segments from $\boldsymbol{0}$ to $\boldsymbol{x}$, so $\mathbb{R}^3$ consists of vectors; and again, we may not always distinguish $\boldsymbol{x}$ from the translate whose initial point is $\boldsymbol{a} = (a_1, a_2, a_3)$ and terminal point $(x_1 + a_1, x_2 + a_2, x_3 + a_3)$ (Figure 17). A double use of Pythagoras's theorem, for the triangles with vertices $\boldsymbol{0}, (0, x_2, 0), (x_1, x_2, 0)$, and vertices $\boldsymbol{0}, (x_1, x_2, 0), (x_1, x_2, x_3)$ leads to the formula for the length $|\boldsymbol{x}|$ of vector $\boldsymbol{x}$ (Figure 18):

$$|\boldsymbol{x}| = (x_1^2 + x_2^2 + x_3^2)^{1/2}. \tag{8}$$

For, in the second triangle, $|\boldsymbol{x}|^2 = ((x_1^2 + x_2^2)^{1/2})^2 + x_3^2 = x_1^2 + x_2^2 + x_3^2$.

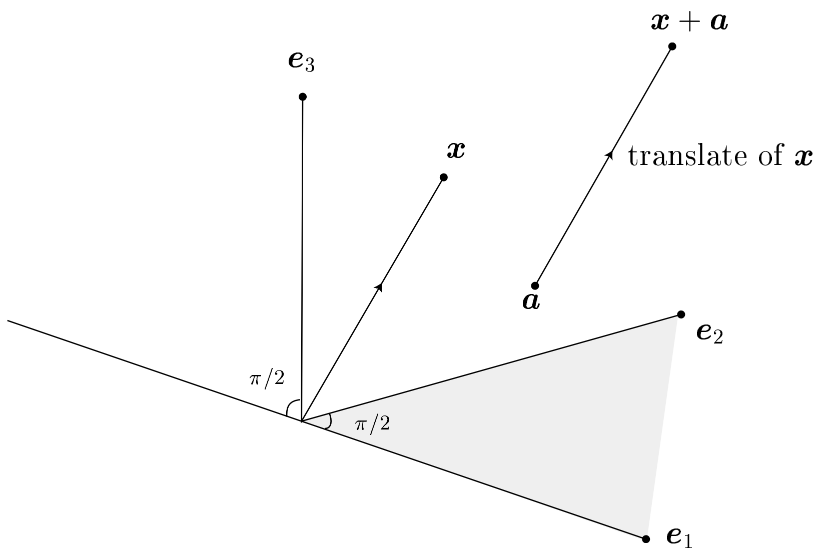

Figure 17. Translate of a vector in $\mathbb{R}^3$.

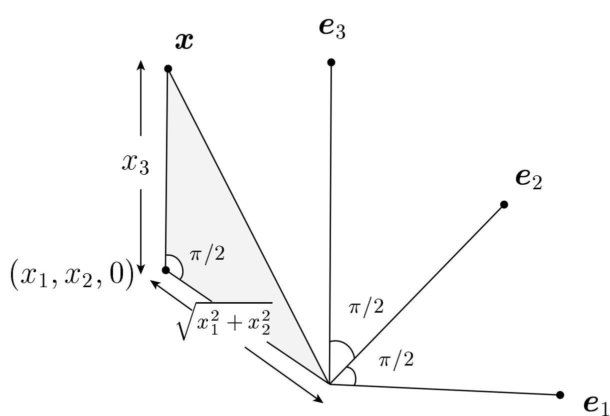

Figure 18. Double application of Pythagoras's theorem.

Definition 5 The **sum** of $\boldsymbol{a} = (a_1, a_2, a_3)$ and $\boldsymbol{b} = (b_1, b_2, b_3)$ is

$$\boldsymbol{a} + \boldsymbol{b} = (a_1 + b_1, a_2 + b_2, a_3 + b_3).$$

Geometrically, $\boldsymbol{a}+\boldsymbol{b}$ is at the terminal point of the vector $\boldsymbol{b}$ if $\boldsymbol{b}$ is placed with initial point at $\boldsymbol{a}$. Alternatively, the description of $\boldsymbol{a}+\boldsymbol{b}$, as the fourth vertex of the parallelogram with vertices $\mathbf{0}, \boldsymbol{a}, \boldsymbol{b}$, that we used in $\mathbb{R}^2$, is valid. This is often referred to as the **parallelogram law of addition**.

Subtraction of $\boldsymbol{z}$ from $\boldsymbol{y}$ is defined by

$$(\boldsymbol{y} - \boldsymbol{z}) + \boldsymbol{z} = \boldsymbol{y}.$$

Thus

$$\boldsymbol{y} - \boldsymbol{z} = (y_1 - z_1, y_2 - z_2, y_3 - z_3).$$

It is clear that the distance from $\boldsymbol{z}$ to $\boldsymbol{y}$ is $|\boldsymbol{y} - \boldsymbol{z}|$.

Definition 6 The **scalar product** $c\boldsymbol{x}$, where c is real and $\boldsymbol{x}$ is in $\mathbb{R}^3$, is

$$c\boldsymbol{x} = (cx_1, cx_2, cx_3).$$

We can, as before, describe $c\boldsymbol{x}$ as the vector pointing in the same direction as $\boldsymbol{x}$, but with length c times that of $\boldsymbol{x}$, if $c > 0$; or the opposite direction, with length $-c$ times that of $\boldsymbol{x}$, if $c < 0$. This depends on the formula

$$|c\boldsymbol{x}| = |c|\ |\boldsymbol{x}|$$

which we can get in a very similar way to the proof in $\mathbb{R}^2$:

$$|c\boldsymbol{x}| = ((cx_1)^2 + (cx_2)^2 + (cx_3)^2)^{1/2} = |c|(x_1^2 + x_2^2 + x_3^2)^{1/2} = |c|\ |\boldsymbol{x}|.$$

You can probably guess the formula that we use to define inner product. It is

Definition 7 The **inner product** of $\boldsymbol{u}$ and $\boldsymbol{v}$ is

$$\boldsymbol{u} \cdot \boldsymbol{v} = u_1v_1 + u_2v_2 + u_3v_3.$$

This is consistent with Definition 4 if we think of $\mathbb{R}^2$ as being the same as the set of all $(x_1, x_2, 0)$ (or, the set of all $x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2$) where x_1 and x_2 are real.

Note that the equation (4) from $\mathbb{R}^2$,

$$|\boldsymbol{u}| = (\boldsymbol{u} \cdot \boldsymbol{u})^{1/2},$$

holds good in $\mathbb{R}^3$.

The angle a between nonzero vectors $\boldsymbol{u}, \boldsymbol{v}$ can be defined by drawing a plane that contains $\boldsymbol{0}, \boldsymbol{u}, \boldsymbol{v}$ and measuring a (between 0 and π) in that plane.

Proposition 5 *We have*

$$\boldsymbol{u} \cdot \boldsymbol{v} = |\boldsymbol{u}|\ |\boldsymbol{v}| \cos a.$$

Proof. We may repeat the proof of Proposition 3 verbatim.

'Verbatim' means 'word for word.' When you want to use for yourself a proof along these lines, bear in mind that students regularly arrive at wrong conclusions this way. If you try it with Proposition 4, you arrive at the false conclusion that

$$\boldsymbol{u} \cdot \boldsymbol{x} = \boldsymbol{u} \cdot \boldsymbol{a}$$

is the equation of a line in $\mathbb{R}^3$ (it is a plane; see below!). Thus you have to be very careful that arguments still work in a new context.

As in $\mathbb{R}^2$, for *nonzero* vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ the relation

$$\boldsymbol{u} \cdot \boldsymbol{v} = 0 \tag{9}$$

is equivalent to $\boldsymbol{u}, \boldsymbol{v}$ being perpendicular. We say $\boldsymbol{u}$ and $\boldsymbol{v}$ are **orthogonal** if (9) holds.

Example 8 The angle between the vectors $(1, 7, b)$ and $(-2, 2, 1)$ is $a = \cos^{-1}(1/3)$. Find b.

Solution. We know that

$$\begin{aligned}(1, 7, b) \cdot (-2, 2, 1) &= |(1, 7, b)|\ |(-2, 2, 1)| \cos a \\ &= |(1, 7, b)|\ |(-2, 2, 1)| 1/3.\end{aligned}$$

Thus

$$\begin{aligned}-2 + 14 + b &= (1^2 + 7^2 + b^2)^{1/2}(2^2 + 2^2 + 1)^{1/2} 1/3; \\ b + 12 &= (b^2 + 50)^{1/2}.\end{aligned}$$

If we square both sides and cancel b^2 we find that

$$24b + 144 = 50;\ b = -47/12.$$